§SW1 Toward a wavelet, given our scaling function of compact support

We made a scaling function, starting from a "low-pass filter." We couldn't call our "scaling function" one, really, unitil we showed it gave us an MRA. We'll do a similar thing now. To do so we recall that an MRA consists of a *nested* sequence of closed subspaces V_j of $L^2(\mathbb{R})$, $V_j \subseteq V_{j+1}$. We recall too that a wavelet gives us an orthonormal basis of $L^2(\mathbb{R})$. We will start with $V_0 \subseteq V_1$. Thus there is a part of V_1 that is not contained in V_0. We want to know more precisely what that part is. The notion of "dropping a perpendicular" comes into its own here. We find that it's useful to deal with a situation such as $V_j \subseteq V_{j+1}$ by *projecting* V_{j+1} perpendicularly onto V_j. We have an operator, $P_j := P_{V_j}$, that does this for us. In addition, it tells us something about the "remainder:" $y - P_j y$ is perpendicular to V_j. Thus every element v of V_{j+1} can be written as $v = v_1 + v_0$, where $v_0 \in V_j$, $v_1 \in V_{j+1}$ and $v_1 \perp v_0$. This is how we proceed: we define W_j to be the subspace of V_{j+1} that is perpendicular to V_j and then write V_{j+1} as $W_j \oplus V_j$. These bits, W_j, defined (using W_0) in the same way that the V_j were defined in terms of V_0, turn out to be perpendicular to each other and when "put together" they give a way to split $L^2(\mathbb{R})$ into similar mutually perpendicular pieces. The picture will clear as we proceed.

We seek to obtain a wavelet from the scaling function we have "constructed." We will still assume we have a trigonometric polynomial $m_o(\xi)$ in the rôle of low-pass filter, and that $(h0)$–$(h5)$ are true. Much of what we do won't need all those assumptions, but it's more convenient for now to have them all.

§SW2 What $f \in V_1$ are orthogonal to V_0?

We have $V_0 \subseteq V_1$. We seek the $f \in V_1$ that are perpendicular to V_0. We want to find an orthonormal basis for them, namely an orthonormal basis for $V_0^\perp \cap V_1$, the orthogonal complement of V_0 as a subspace of V_1.

To begin we want to know how to tell, using the Fourier transform, which $f \in L^2$ are in V_1. We know that f is in V_1 if and only if $f(t) = \sum c_n \sqrt{2}\varphi(2t-n)$, where $\sum |c_n|^2 < \infty$. Thus

$$(S1) \qquad \widehat{f}(\xi) = \sum c_n e^{-i\xi n/2} \frac{1}{\sqrt{2}}\,\widehat{\varphi}(\xi/2) = \left(\sum \frac{c_n}{\sqrt{2}} e^{-i\xi n/2}\right)\widehat{\varphi}(\xi/2) =: p(\xi/2)\,\widehat{\varphi}(\xi/2),$$

where $p(\xi)$ is a 2π-periodic function in $L^2(\mathbb{T})$. Explanation: the series for f converges in the L^2 sense. As the Fourier transform is continuous on L^2, and the series converges in L^2, so does the series of Fourier transforms of the terms. Then we factor out the exponentials that arose because of translations by half-integers.

Now that we have a way to identify functions f in V_1, we ask which of them are perpendicular to V_0. We'll give the answer, then check it.
Answer 1: Those f with $\widehat{f}(\xi) = \eta(\xi/2)\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2)$, where $\eta \in L^2(\mathbb{T})$, $\eta(\xi+\pi) = -\eta(\xi)$.

We'll show the function $p(\xi/2)$ that appeared in $(S1)$ has the *added* property $p(\xi+\pi) = -p(\xi)$.

If $f \in V_0^\perp \cap V_1$ then $\langle f, \varphi(t-n)\rangle_{dt} = 0$ for all $n \in \mathbb{Z}$. On the other hand, if $\langle f, \varphi(t-n)\rangle_{dt} = 0$ for all $n \in \mathbb{Z}$, then f is perpendicular to any linear combination of the $\varphi(t-n)$. But then f is perpendicular to the closure of the span of the $\varphi(t-n)$, which is precisely V_0. Hence all we have to do is find the members f of V_1 that are perpendicular to each $\varphi(t-n)$. As we have learned by now, we'll use the Fourier transform. By $(S1)$,

$$(S2) \qquad \langle f, \varphi(t-n)\rangle_{dt} = \int f(t)\overline{\varphi(t-n)}\,dt = \frac{1}{2\pi}\int \widehat{f}(\xi)\overline{e^{-in\xi}\widehat{\varphi}(\xi)}\,d\xi = \frac{1}{2\pi}\int p(\xi/2)\,\widehat{\varphi}(\xi/2)\overline{e^{-in\xi}\widehat{\varphi}(\xi)}\,d\xi.$$

We want the last integral in $(S2)$ to be zero for all n. By the Fourier Scaling equation, $\widehat{\varphi}(\xi) = m_o(\xi/2)\widehat{\varphi}(\xi/2)$ so we can insert this into the last integral in $(S2)$, move $e^{-in\xi}$ out from under the bar, and get

$$\int f(t)\overline{\varphi(t-n)}\,dt = \frac{1}{2\pi}\int e^{in\xi}p(\xi/2)\overline{m_o(\xi/2)}\,|\widehat{\varphi}(\xi/2)|^2\,d\xi.$$

We want this to be zero for every n. Once again we change variables, replacing ξ by 2ξ and *periodize* (see the Fifth Question part of the notes "From low-pass filter to scaling function" for details) which gives

$$(S2.5) \qquad \int f(t)\overline{\varphi(t-n)}\,dt = \frac{1}{\pi}\int_0^{2\pi} e^{i2n\xi}p(\xi)\overline{m_o(\xi)}\,d\xi. \text{ Which } p(\xi) \text{ make this zero for all } n?$$

($S2.6$) **Exercise:** *Carry out the periodization that leads to* ($S2.5$).

When we used periodization before, we ended with the exponential $e^{in\xi}$ instead of $e^{i2n\xi}$, which meant we wanted the rest of the integrand back then to be a constant. We would like to use the same idea here, so we need to get rid of the factor 2 in the exponent. This is done next; we'll split $[0,\, 2\pi)$ into its two halves, and make a change of variables that doubles each interval (and moves the second one back to start at 0).

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi} e^{i2n\xi}p(\xi)\overline{m_o(\xi)}\,d\xi &= \frac{1}{\pi}\int_0^{\pi} e^{i2n\xi}p(\xi)\overline{m_o(\xi)}\,d\xi + \frac{1}{\pi}\int_\pi^{2\pi} e^{i2n\xi}p(\xi)\overline{m_o(\xi)}\,d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi}p(\xi/2)\overline{m_o(\xi/2)}\,d\xi + \frac{1}{\pi}\int_0^{\pi} e^{i2n\xi}p(\xi+\pi)\overline{m_o(\xi+\pi)}\,d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi}p(\xi/2)\overline{m_o(\xi/2)}\,d\xi + \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi}p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)}\,d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi}\left(p(\xi/2)\overline{m_o(\xi/2)} + p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)}\right)d\xi.\end{aligned}$$

Putting this back into ($S2$) gives us

$$\int f(t)\overline{\varphi(t-n)}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{in\xi}\left(p(\xi/2)\overline{m_o(\xi/2)} + p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)}\right)d\xi,$$

which is *now* zero for *every* n if and only if

$$p(\xi/2)\overline{m_o(\xi/2)} + p(\xi/2+\pi)\overline{m_o(\xi/2+\pi)} = 0 \text{ a.e.},$$

or, simplifying by getting rid of the divisions by 2, we have found that

($S3$) If $f \in V_1$, then $f \perp V_0 \iff p(\xi)\overline{m_o(\xi)} + p(\xi+\pi)\overline{m_o(\xi+\pi)} = 0$ a.e., where $\widehat{f}(\xi) = p(\xi/2)\widehat{\varphi}(\xi/2)$.

The equation $p(\xi)\overline{m_o(\xi)} + p(\xi+\pi)\overline{m_o(\xi+\pi)} = 0$ has the *form* $\langle w,\, z\rangle = w_1\overline{z_1} + w_2\overline{z_2} = 0$, where $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a non-zero vector in $\mathbb{C}^2$, $w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is also in $\mathbb{C}^2$ and $\langle w,\, z\rangle$ denotes their inner product in $\mathbb{C}^2$. In other words we want $w \perp z$ in $\mathbb{C}^2$. It is an exercise in algebra to show that we must have $w = \zeta\begin{pmatrix} -\overline{z_2} \\ \overline{z_1} \end{pmatrix}$, where ζ is a (complex) constant of proportionality. To return to the matter at hand, ($S3$), we have an equation at each point ξ, so we need to have a *function* $\zeta(\xi)$ such that

$$\begin{pmatrix} p(\xi) \\ p(\xi+\pi) \end{pmatrix} = \zeta(\xi)\begin{pmatrix} -\overline{m_o(\xi+\pi)} \\ \overline{m_o(\xi)} \end{pmatrix} \text{ a.e.}$$

This is really two equations:

$$\begin{aligned} p(\xi) &= -\,\zeta(\xi)\overline{m_o(\xi+\pi)} \text{ a.e.,} \\ &\text{and} \\ p(\xi+\pi) &= \zeta(\xi)\overline{m_o(\xi)} \text{ a.e.} \end{aligned}$$

Since these equations are to hold for almost every ξ, we can replace ξ by $\xi+\pi$ in both equations, and then take advantage of 2π-periodicity. When we do so, we get

$$\begin{aligned} p(\xi+\pi) &= -\,\zeta(\xi+\pi)\overline{m_o(\xi)} \text{ a.e.,} \\ &\text{and} \\ p(\xi) &= \zeta(\xi+\pi)\overline{m_o(\xi+\pi)} \text{ a.e.} \end{aligned}$$

Therefore (using the second equation in the first pair and the first one in the second pair)

$$\zeta(\xi)\overline{m_o(\xi)} = p(\xi+\pi) = -\zeta(\xi+\pi)\overline{m_o(\xi)} \text{ a.e.}$$

In our special case, $m_o(\xi) \neq 0$ a.e. because m_o is a trigonometric polynomial so

$$\zeta(\xi+\pi) = -\zeta(\xi) \text{ a.e.}$$

Therefore,

$$p(\xi) = -\zeta(\xi)\overline{m_o(\xi+\pi)},$$

where $\zeta(\xi)$ is 2π-periodic and has the property that $\zeta(\xi+\pi) = -\zeta(\xi)$ a.e. Moreover, by $(h5)$ $m_o(\xi) \neq 0$ if $|\xi| \leq \pi/2$, so $|m_o(\xi)| \geq A$ for some positive A if $|\xi| \leq \pi/2$. Thus $\zeta(\xi)$ is square-integrable on $[-\pi/2, \pi/2]$. But then, since $p(\xi) \in L^2(\mathbb{T})$ and $\zeta(\xi+\pi) = -\zeta(\xi)$ a.e., it is true that $\zeta(\xi) \in L^2(\mathbb{T})$. Thus, with $\eta(\xi) := -\zeta(\xi)$, so that now $\eta \in L^2(\mathbb{T})$, we have

$$(S4) \qquad p(\xi) = \eta(\xi)\overline{m_o(\xi+\pi)}, \text{ where } \eta(\xi) \in L^2(\mathbb{T}) \text{ and } \eta(\xi+\pi) = -\eta(\xi) \text{ a.e.}$$

This completes the proof of our Answer 1 to the question: **What functions $f \in V_1$ are orthonormal to V_0?**

We need a better answer. The meaning of $\eta(\xi+\pi) = -\eta(\xi)$ a.e. will appear on comparing the two series in

$$\eta(\xi+\pi) = \sum \eta_n e^{in(\xi+\pi)} = -\eta(\xi) = -\sum \eta_n e^{in\xi},$$

so that

$$\sum (-1)^n \eta_n e^{in\xi} = -\sum \eta_n e^{in\xi} \text{ a.e.,}$$

which means, *by the uniqueness of Fourier coefficients*, that $(-1)^n\eta_n = -\eta_n$ for all n. Thus $\eta_{2k} = 0$ for all k and we can now give an alternate answer to the original question. For now

$$\eta(\xi) = \sum_k \eta_{2k+1} e^{i(2k+1)\xi} = e^{i\xi}\sum_k \eta_{2k+1} e^{i2k\xi} =: e^{i\xi}H(2\xi), \text{ where } H(\xi) \in L^2(\mathbb{T})$$

so that $H(\xi) = \sum_k \eta_{2k+1}e^{ik\xi} =: \sum_k H_k e^{ik\xi}$. I.e., $H_k := \eta_{2k+1}$, all k. Thus $\eta(\xi) = e^{i\xi}H(2\xi)$, so

Answer 2: $f \in V_1 \cap V_0^{\perp}$ if and only if

$$(S5) \qquad \widehat{f}(\xi) = \eta(\xi/2)\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2) = H(\xi)e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2), \text{ where } H \in L^2(\mathbb{T}).$$

A wavelet for φ, defined via the Fourier transform

When we stare at it awhile it will dawn on us that $(S5)$ is a formula that stands for an L^2 series expressing $f(t)$ in terms of the integer translates of the function whose Fourier transform is $e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2)$. To check this, let's *define* a function $\psi(t)$ by saying that its Fourier transform is

$$(S6\text{: wavelet-hat}) \qquad \widehat{\psi}(\xi) := e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2).$$

We can make this resemble the Fourier Scaling equation,

$$\widehat{\varphi}(\xi) = m_o(\xi/2)\widehat{\varphi}(\xi/2),$$

By *defining*

$$(S7\text{: hi-pass filter?}) \qquad m_1(\xi) = e^{i\xi}\overline{m_o(\xi+\pi)},$$

and then we'll have $\widehat{\psi}(\xi) = m_1(\xi/2)\widehat{\varphi}(\xi/2)$. We have a very important question to ask now.

Is the set $\{\psi(t-n) : n \in \mathbb{Z}\}$ an orthonormal set?

In order to show that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set we proved that

$$G(\xi) := \sum_n |\widehat{\varphi}(\xi + 2\pi n)|^2 = 1 \text{ a.e.}$$

We can do the same for $\{\psi(t-n) : n \in \mathbb{Z}\}$. All we need to do is verify that

$$\sum_n |\widehat{\psi}(\xi + 2\pi n)|^2 = 1 \text{ a.e.}$$

We have

$$|\widehat{\psi}(\xi)|^2 = |m_1(\xi/2)|^2 |\widehat{\varphi}(\xi/2)|^2 = |e^{i\xi}\overline{m_o(\xi/2+\pi)}|^2 |\widehat{\varphi}(\xi/2)|^2 = |m_o(\xi/2+\pi)|^2 |\widehat{\varphi}(\xi/2)|^2.$$

Thus

$$\sum_n |\widehat{\psi}(\xi+2\pi n)|^2 = \sum_n |m_o(\xi/2+\pi n+\pi)|^2 |\widehat{\varphi}(\xi/2+\pi n)|^2.$$

We divide the n's into "evens" and "odds," as we have done before. This gives us: $\sum_n |\widehat{\psi}(\xi + 2\pi n)|^2$ is equal to

$$\begin{aligned}
&\sum_n |m_o(\xi/2 + \pi 2n + \pi)|^2 |\widehat{\varphi}(\xi/2 + \pi 2n)|^2 + \sum_n |m_o(\xi/2 + \pi(2n+1) + \pi)|^2 |\widehat{\varphi}(\xi/2 + \pi(2n+1))|^2 \\
&= \sum_n |m_o(\xi/2 + \pi)|^2 |\widehat{\varphi}(\xi/2 + 2\pi n)|^2 + \sum_n |m_o(\xi/2)|^2 |\widehat{\varphi}(\xi/2 + \pi + 2\pi n)|^2 \\
&= |m_o(\xi/2 + \pi)|^2 G(\xi/2) + |m_o(\xi/2)|^2 G(\xi/2 + \pi) \\
&= |m_o(\xi/2 + \pi)|^2 + |m_o(\xi/2)|^2 = 1 \text{ a.e.}
\end{aligned}$$

We have shown that $\{\psi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set. We plan to show that $V_1 = W_0 \oplus V_0$.

$(S7.5)$ **Exercise:** *Verify that* $\{\psi(t-n) : n \in \mathbb{Z}\} \cup \{\varphi(t-n) : n \in \mathbb{Z}\}$ *is an orthonormal set contained in* V_1.

Let us return to $(S5)$ now and write

$$\widehat{f}(\xi) = H(\xi)e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2) = H(\xi)\widehat{\psi}(\xi) = \sum_k H_k e^{ik\xi}\widehat{\psi}(\xi) = \sum_k H_k \widehat{\left[\psi(t+k)\right]}(\xi) = \widehat{\left[\sum_k H_k \psi(t+k)\right]}(\xi).$$

In the last sequence of equations the truth of the last equality depended on knowing that $\{\psi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set. This gives us (let's let $F_k := H_{-k}$):

Answer 3: $f \in V_1 \cap V_0^\perp$ if and only if

$$f(t) = \sum_k F_k \psi(t-k), \quad \text{where} \quad \sum_k |F_k|^2 < \infty.$$

Answer 3 leads to the following definition, which can be regarded as our **Final Answer** to the original question:

$(S8)$ $$W_0 := V_1 \cap V_0^\perp = \overline{\operatorname{span}\{\psi(t-k) : k \in \mathbb{Z}\}} \text{ and } W_j := \{f(2^j t) : f(t) \in W_0\}, \text{ for all } j \in \mathbb{Z}.$$

To show that $V_1 = W_0 + V_0$ all we need to show is that if $f \in V_1$ then there exists $f_1 \in V_0$ and $f_2 \in W_0$ such that $f = f_1 + f_2$. We already know that $W_0 \perp V_0$. We define $f_1 := P_0 f$, the orthogonal projection of f on V_0. Then $f - f_1 \perp V_0$. Since we know that $f \in V_1$, we can define $f_2 := f - f_1$. Then $f_2 \in V_1$ and $f_2 \perp V_0$, so $f_2 \in V_1 \cap V_0^\perp = W_0$. This is what we needed to show. To show that $V_1 = W_0 \oplus V_0$ we need to show that $W_0 \perp V_0$. This follows immediately from the definition of W_0. Suggestion: Go back and read the opening paragraph now.

§SW3 Digressions: our "wavelet," expressed in terms of the scaling function

We know that $\psi(t) \in V_1$. Our goal right now is to find out more about the coefficients ψ_n in the L^2 series

$$\psi(t) = \sum \psi_n \sqrt{2}\varphi(2t-n).$$

We can use the same idea that led to $(S6)$. We know by $(S6)$ that

$$(S9)\qquad \begin{aligned}\widehat{\psi}(\xi) =&e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2)\\ =&e^{i\xi/2}\overline{\sum_n \frac{h_n}{\sqrt{2}}e^{-in(\xi/2+\pi)}}\widehat{\varphi}(\xi/2)\\ =&e^{i\xi/2}\sum_n \frac{\overline{h_n}}{\sqrt{2}}e^{in\xi/2}e^{in\pi}\widehat{\varphi}(\xi/2)\\ =&\sum_n \frac{\overline{(-1)^n h_n}}{\sqrt{2}}e^{i(n+1)\xi/2}\widehat{\varphi}(\xi/2)\\ =&\sum_n \frac{\overline{-(-1)^n h_{n-1}}}{\sqrt{2}}e^{in\xi/2}\widehat{\varphi}(\xi/2)\\ =&\sum_n \frac{\overline{-(-1)^n h_{-n-1}}}{\sqrt{2}}e^{-in\xi/2}\widehat{\varphi}(\xi/2)=m_1(\xi/2)\widehat{\varphi}(\xi/2).\end{aligned}$$

We're not quite done; so far, what we have is a formula for $m_1(\xi)$:

$$m_1(\xi)=\sum_n \frac{\overline{-(-1)^n h_{-n-1}}}{\sqrt{2}}e^{-in\xi}=:\sum_n \frac{h_n^{(1)}}{\sqrt{2}}e^{-in\xi},$$

where $h_n^{(1)}=(-1)^{n+1}\overline{h_{-n-1}}$. To continue with $(S9)$, we have

$$\begin{aligned}\widehat{\psi}(\xi) =&\sum_n \frac{\overline{-(-1)^n h_{-n-1}}}{\sqrt{2}}e^{-in\xi/2}\widehat{\varphi}(\xi/2)\\ =&\sum_n h_n^{(1)}e^{-in\xi/2}\frac{1}{\sqrt{2}}\widehat{\varphi}(\xi/2)\\ =&\sum_n h_n^{(1)}\widehat{\left[\sqrt{2}\varphi(2t-n)\right]}(\xi)\\ =&\widehat{\left[\sum_n h_n^{(1)}\sqrt{2}\varphi(2t-n)\right]}(\xi),\end{aligned}$$

so that we have

$$(S9.5)\qquad \psi(t)=\sum_n h_n^{(1)}\sqrt{2}\varphi(2t-n),\quad \text{where}\quad h_n^{(1)}=(-1)^{n+1}\overline{h_{-n-1}}=\left\langle \psi(t),\,\sqrt{2}\varphi(2t-n)\right\rangle_{dt}.$$

Another digression: The support of ψ in terms of that of φ

We know that $\varphi(t)=0$ unless $M_1\le t\le M_2$, where M_1 and M_2 are the integers such that $h_n=0$ unless $M_1\le n\le M_2$. Then since $h_n^{(1)}=(-1)^{n+1}\overline{h_{-n-1}}$, in $(S9.5)$ we have to have $M_1\le -n-1\le M_2$, so $-M_2-1\le n\le -M_1-1$. Then when $n=-M_2-1$, we have $\varphi(2t-n)=\varphi(2t+M_2+1)=0$ unless $M_1\le 2t+M_2+1\le M_2$, or $(M_1-M_2-1)/2\le t$. When n is as large as possible, $n=-M_1-1$, a similar calculation yields $t\le (M_2-M_1-1)/2$, so *this* wavelet $\psi(t)$ is zero unless $(M_1-M_2-1)/2\le t\le (M_2-M_1-1)/2$. We notice that the length of this supporting interval is the same as that for φ, namely M_2-M_1. However, its location is different unless $M_1+M_2=-1$. We can always translate this $\psi(t)$ by an integer, and call *that* translate a wavelet too. If we translate by N, i.e. replace $\psi(t)$ by $\psi(t-N)$, the new support interval will be $(M_1-M_2+2N-1)/2\le t\le (M_2-M_1+2N-1)/2$. In particular we can arrange for the support interval to be the same as that for φ if $N=(M_1+M_2+1)/2$.

$(S9.6)$ **Exercise:** *Show that, if* $h_{M_1}\ne 0\ne h_{M_2}$, *then* M_1+M_2+1 *is even.*

§SW4 To resume: We need to know more about what $V_1 = W_0 \oplus V_0$ means

We want to know more detail; in particular, we want a *formula* for expressing functions in V_1 in terms of the *integer* translates of the wavelet and of the scaling function. Of special importance: finding out how the coefficients in the two formulas in $(S10.00)$ (just below) are related.

One way we can be sure that $V_1 = W_0 \oplus V_0$ is to show that $\{\psi(t-n) : n \in \mathbb{Z}\} \cup \{\varphi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal basis of V_1. If we can do that, we'll know that every $f \in V_1$ can be written in two ways:

$$(S10.00) \qquad f = \sum_k \langle f,\, \varphi_{1k}\rangle\, \varphi_{1k} = \sum_k \langle f,\, \psi_{0k}\rangle\, \psi_{0k} + \sum_k \langle f,\, \varphi_{0k}\rangle\, \varphi_{0k}.$$

Our objective next is to show that the coefficients in the last sum, of two series, can be gotten by performing a "matrix multiplication" on the coefficients in the first sum, $\sum_k \langle f,\, \varphi_{1k}\rangle\, \varphi_{1k}$. To do this we will find a series for φ_{10} and a series for φ_{11}, using the ψ_{0k} and the φ_{0k}. From the point of view of the functions ψ_{0k} and the φ_{0k}, φ_{10} and φ_{11} are very, very different, because we *cannot* translate the functions ψ_{0k} and the φ_{0k} by $1/2$ and expect them to remain in W_0 and V_0, respectively!

$(S10.01)$ **Exercise:** *Show that* $\{\psi(t-n) : n \in \mathbb{Z}\} \cup \{\varphi(t-n) : n \in \mathbb{Z}\}$ *is an orthonormal basis of* V_1.

The Exercise shows that

$$(S10.02) \qquad \varphi_{10}(t) := \sqrt{2}\varphi(2t) \text{ is equal to } \sum_k \langle \varphi_{10},\, \psi(s-k)\rangle_{ds}\psi(t-k) + \sum_k \langle \varphi_{10},\, \varphi(s-k)\rangle_{ds}\varphi(t-k)$$

and that

$$(S10.03) \qquad \varphi_{11}(t) := \sqrt{2}\varphi(2t-1) \text{ is equal to } \sum_k \langle \varphi_{11},\, \psi(s-k)\rangle_{ds}\psi(t-k) + \sum_k \langle \varphi_{11},\, \varphi(s-k)\rangle_{ds}\varphi(t-k).$$

As always each series is interpreted as an L^2 series. We have to show both equations, because the functions φ and ψ can only be translated by whole integer steps (in order to stay in V_0 and W_0), not the half-integer steps that work in V_1. We will find the coefficients in these equations and then use the equations and translations to find equations for every φ_{1n}.

We need the Scaling Equation, page 1 of the MRA notes (we used $h(n)$ there): $\varphi(t) = \sum_n h_n \sqrt{2}\, \varphi(2t-n)$, as well as $(S9.5)$. Then $\langle \varphi_{10},\, \psi(s-k)\rangle_{ds}$ is given by

$$(S10.04) \qquad \int \sqrt{2}\varphi(2s)\overline{\psi(s-k)}\, ds = \overline{\int \overline{\sqrt{2}\varphi(2s)}\psi(s-k)\, ds} = \overline{\int \psi(s)\overline{\sqrt{2}\varphi(2s-(-2k))}\, ds} =: \overline{h^{(1)}_{-2k}} = -h_{2k-1}$$

and $\langle \varphi_{10},\, \varphi(s-k)\rangle_{ds}$ is given by

$$(S10.05) \qquad \int \sqrt{2}\varphi(2s)\overline{\varphi(s-k)}\, ds = \overline{\int \overline{\sqrt{2}\varphi(2s)}\varphi(s-k)\, ds} = \overline{\int \varphi(s)\overline{\sqrt{2}\varphi(2s-(-2k))}\, ds} = \overline{h_{-2k}}.$$

To find the coeffecients for φ_{11} we substitute $\sqrt{2}\varphi(2s-1)$ for $\sqrt{2}\varphi(2s)$ in $(S10.04)$ and $(S10.05)$.

$(S10.06)$ **Exercise:** *Verify that*

$$\varphi_{11} = \sum_k h_{2k-2}\, \psi_{0k} + \sum_k \overline{h_{-2k+1}}\, \varphi_{0k} \quad \textit{and that} \quad \varphi_{10} = \sum_k -h_{2k-1}\, \psi_{0k} + \sum_k \overline{h_{-2k}}\, \varphi_{0k}.$$

Since we can translate either of these two expressions for φ_{10} and φ_{11} to the right or left by any integer (we translate the whole series in each case) we can see that every φ_{1n} is expressible similarly.

$(S10.07)$ **Exercise:** *Show that*

$$\varphi_{1,2n}(t) = \sum_k \overline{h_{-2k+2n}}\varphi(t-k) + \sum_k -h_{2k-2n-1}\psi(t-k) \textit{ and } \varphi_{1,2n+1}(t) = \sum_k \overline{h_{-2k+2n+1}}\varphi(t-k) + \sum_k h_{2k-2n-2}\psi(t-k).$$

Shorter versions: $\varphi_{1,2n} = \sum_k \overline{h_{-2k+2n}}\varphi_{0k} + \sum_k -h_{2k-2n-1}\psi_{0k}$ and $\varphi_{1,2n+1} = \sum_k \overline{h_{-2k+2n+1}}\varphi_{0k} + \sum_k h_{2k-2n-2}\psi_{0k}$.

Converting V_1 coefficients to $W_0 \oplus V_0$ coefficients

Suppose we have $f(t) \in V_1$. Then $f(t) = \sum_m \langle f,\, \varphi_{1m}\rangle\, \varphi_{1m}(t) =: \sum_m c_m \varphi_{1m}(t)$. We can use $(S10.07)$ to write the series for f in terms of φ_{0k} and ψ_{0k}:

$$
\begin{aligned}
P_1 f(t) = f(t) &= \sum_n c_n \varphi_{1n}(t) \quad (\text{where } \ c_n = \langle f,\, \varphi_{1n}\rangle\ ) \\
&= \sum_n c_{2n}\varphi_{1,2n}(t) + \sum_n c_{2n+1}\varphi_{1,2n+1}(t) \\
&= \sum_n c_{2n} \sum_k \overline{h_{-2k+2n}}\varphi(t-k) + \sum_n c_{2n} \sum_k -h_{2k-2n-1}\psi(t-k) \\
&+ \sum_n c_{2n+1} \sum_k \overline{h_{-2k+2n+1}}\varphi(t-k) + \sum_n c_{2n+1} \sum_k h_{2k-2n-2}\psi(t-k) \\
&= \sum_k \left(\sum_n c_{2n}\overline{h_{-2k+2n}} + c_{2n+1}\overline{h_{-2k+2n+1}} \right) \varphi(t-k) \\
&+ \sum_k \left(-\sum_n c_{2n}h_{2k-2n-1} + \sum_n c_{2n+1}h_{2k-2n-2} \right) \psi(t-k). \ \ \text{That is,} \\
f(t) = \sum_k & \left(\sum_n c_n \overline{h_{-2k+n}} \right) \varphi(t-k) + \sum_k \left(\sum_n (-c_{2n}h_{2k-2n-1} + c_{2n+1}h_{2k-2n-2}) \right) \psi(t-k).
\end{aligned}
\tag{S10.5}
$$

A shorter version, with ψ's first:

$$
f = \sum_k \left(\sum_n (-c_{2n}h_{2k-2n-1} + c_{2n+1}h_{2k-2n-2}) \right) \psi_{0k} + \sum_k \left(\sum_n c_n \overline{h_{-2k+n}} \right) \varphi_{0k}. \tag{S10.51}
$$

By the uniqueness of "Fourier coefficients" (coefficients with respect to an orthonormal basis), we have shown (by $(S10.00)$) that for all $k \in \mathbb{Z}$,

$$
\langle f,\, \psi_{0k}\rangle = \sum_n (-c_{2n}h_{2k-2n-1} + c_{2n+1}h_{2k-2n-2}) \quad \text{and} \quad \langle f,\, \varphi_{0k}\rangle = \sum_n c_n \overline{h_{-2k+n}}. \tag{S10.52}
$$

This is a more "practical" way of finding the new coefficients than finding the new inner products would be.

A "matricial" way to express $(S10.52)$

We can write $\sum_n (-c_{2n}h_{2k-2n-1} + c_{2n+1}h_{2k-2n-2}) = (Bc)_k$ and $\sum_n c_n \overline{h_{-2k+n}} = (Ac)_n$, thinking of $c = \{c_n\}$ as a column vector and B and A as matrices. At the definition level, B and A are infinitely large matrices: $B = (B_{kn})$ and $A = (A_{kn})$, where k and n both run from $-\infty$ to ∞. And in $c = \{c_n\}$, n runs from $-\infty$ to ∞. In practice, the fully infinite sizes won't be used. Let's write A first since it's less complicated than B.

$$
A = (A_{kn}) = \left(\ \overline{h_{-2k+n}}\ \right) = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \overline{h_2} & \overline{h_3} & \overline{h_4} & \overline{h_5} & \overline{h_6} & \cdots \\
\cdots & \overline{h_0} & \overline{h_1} & \overline{h_2} & \overline{h_3} & \overline{h_4} & \cdots \\
\cdots & \overline{h_{-2}} & \overline{h_{-1}} & \overline{h_0} & \overline{h_1} & \overline{h_2} & \cdots \\
\cdots & \overline{h_{-4}} & \overline{h_{-3}} & \overline{h_{-2}} & \overline{h_{-1}} & \overline{h_0} & \cdots \\
\cdots & \overline{h_{-6}} & \overline{h_{-5}} & \overline{h_{-4}} & \overline{h_{-3}} & \overline{h_{-2}} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.
$$

Each row contains all the (conjugated) h_n's. But each column contains only every other h_n.

To find B_{kn} we need a sum with c_n terms, not c_{2n} terms and c_{2n+1} terms. In the term $-c_{2n}h_{2k-2n-1}+c_{2n+1}h_{2k-2n-2}$ we can make the $-$ be $(-1)^{2n+1}$ and write $c_{2n+1}h_{2k-2n-2}$ as $(-1)^{2n+1+1}c_{2n+1}h_{2k-2n-1-1}$. This lets us write

$$\sum_n(-c_{2n}h_{2k-2n-1}+c_{2n+1}h_{2k-2n-2})=\sum_n(-1)^{n+1}h_{2k-n-1}c_n=(Bc)_k.$$

Thus

$$B=(B_{kn})=\big(\ (-1)^{n+1}h_{2k-n-1}\ \big)=\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -h_{-3} & h_{-4} & -h_{-5} & h_{-6} & -h_{-7} & \cdots \\ \cdots & -h_{-1} & h_{-2} & -h_{-3} & h_{-4} & -h_{-5} & \cdots \\ \cdots & -h_{1} & h_{0} & -h_{-1} & h_{-2} & -h_{-3} & \cdots \\ \cdots & -h_{3} & h_{2} & -h_{1} & h_{0} & -h_{-1} & \cdots \\ \cdots & -h_{5} & h_{4} & -h_{3} & h_{2} & -h_{1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Here the center row is row zero and the center column is column zero. Once again, the rows contain (a form of) all the h_n's but the columns contain only every other h_n.

Thus in the action of the matrices A and B on the column vector c every h_n appears, in some form, in each $(Ac)_k$ and in each $(Bc)_k$.

§SW5 Converting V_J coefficients to $W_{J-1}\oplus V_{J-1}$ coefficients

Let J be given. If $f\in V_J$, then $f(t)=2^{(J-1)/2}F(2^{J-1}t)$, where $F(t)\in V_1$. Moreover, $\|F\|=\|f\|$ because of the normalizing factor $2^{(J-1)/2}$. Also, if we let $d_n:=\langle F,\,\varphi_{1n}\rangle$ then $d=\{d_n\}$ plays the rôle of c in $(S10.5)$. Thus

$$F(t)=\sum_n\langle F,\,\varphi_{1n}\rangle\,\varphi_{1n}(t)=\sum_n d_n\varphi_{1n}(t)=\sum_k(Bd)_k\,\psi(t-k)+\sum_k(Ad)_k\,\varphi(t-k). \tag{S5.5.1}$$

Now let's look at the d_n more closely. By definition, then a change of variables, then some simplification,

$$\begin{aligned} d_n=\langle F,\,\varphi_{1n}\rangle&=\int F(t)\overline{\sqrt{2}\varphi(2t-n)}\,dt\\ &=\int F(2^{J-1}t)\overline{\sqrt{2}\varphi(2\cdot2^{J-1}t-n)}\,2^{J-1}dt\\ &=\int 2^{(J-1)/2}F(2^{J-1}t)\overline{\sqrt{2}\cdot2^{(J-1)/2}\varphi(2\cdot2^{J-1}t-n)}\,dt\\ &=\int 2^{(J-1)/2}F(2^{J-1}t)\overline{2^{J/2}\varphi(2^Jt-n)}\,dt\\ &=\int f(t)\overline{2^{J/2}\varphi(2^Jt-n)}\,dt=\langle f,\,\varphi_{Jn}\rangle\,. \end{aligned}$$

This says that d_n is the projection coefficient of f with respect to the φ_{Jn}.

Let us put $(S5.5.1)$ back into terms of f by replacing t by $2^{J-1}t$ and then multiplying through by $2^{(J-1)/2}$:

$$\begin{aligned} f(t)=\ &2^{(J-1)/2}F(2^{J-1}t)=\sum_n d_n2^{(J-1)/2}\varphi_{1n}(2^{J-1}t)=\sum_n d_n\varphi_{Jn}(t)\\ =\ &\sum_k(Bd)_k\,2^{(J-1)/2}\psi(2^{J-1}t-k)+\sum_k(Ad)_k\,2^{(J-1)/2}\varphi(2^{J-1}t-k)\\ =\ &\sum_k(Bd)_k\,\psi_{J-1,k}(t)+\sum_k(Ad)_k\,\varphi_{J-1,k}(t)\in W_{J-1}\oplus V_{J-1}. \end{aligned} \tag{S5.5.2}$$

What if f is a "general" function in $L^2(\mathbb{R})$? Then $(S5.5.2)$ changes to

$$\begin{aligned} P_Jf(t)&=\sum_n\langle f,\,\varphi_{Jn}\rangle\,\varphi_{Jn}(t)=:\sum_n d_n\varphi_{Jn}(t)\\ &=\sum_k(Bd)_k\,\psi_{J-1,k}(t)+\sum_k(Ad)_k\,\varphi_{J-1,k}(t)\in W_{J-1}\oplus V_{J-1}. \end{aligned} \tag{S5.5.3}$$

Thus, **if we start with the projection coefficients of $f(t)$ on V_J, exactly the same rule for splitting $f \in V_1$ into $Q_0 f + P_0 f \in W_0 \oplus V_0$ is used for splitting $P_J f \in V_J$ into $Q_{J-1}f + P_{J-1}f \in W_{J-1} \oplus V_{J-1}$.**

This gives us a shorter and a very short version of $(S5.5.3)$:

$$P_J f = \sum_k (Bd)_k\, \psi_{J-1,k} + \sum_k (Ad)_k\, \varphi_{J-1,k} = Q_{J-1}f + P_{J-1}f. \tag{$S5.5.4$}$$

We can start over again, working on $P_{J-1}f$. Then

$$P_{J-1} f = \sum_k (B[Ad])_k\, \psi_{J-2,k} + \sum_k (A[Ad])_k\, \varphi_{J-1,k} = Q_{J-2}f + P_{J-2}f. \tag{$S5.5.5$}$$

The splitting rule is the same. The coefficients that are used in the splitting are different: d was replaced by Ad. At each succeeding step the power of A increases by one. But we don't compute powers of A. We just keep applying A to the new column vector calculated at the previous stage. Somewhere we'll probably keep d, then only keep (the recursively calculated) $A^m d$.

Remark: Note that here J can be any integer. It does not have to be at least one.

§SW6 Next, the verification that we do have a wavelet

We will call $\psi(t)$ *a wavelet* associated with the scaling function $\varphi(t)$. The term "wavelet" is applied here a little prematurely. We need to show that the family $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. We have shown that $\{\psi_{0k} : k \in \mathbb{Z}\} = \{\psi(t-k) : k \in \mathbb{Z}\}$ is orthonormal. First we'll show that the whole family $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is orthonormal. Finally we will show that $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ by showing that the only function in $L^2(\mathbb{R})$ that is orthogonal to the whole family is the zero function.

We have seen that $V_1 = V_0 \oplus W_0$. Since each space V_j can be obtained from V_0 by a dilation depending on j, we can use the same dilation to define a space W_j. Then $\{\psi_{jk} : k \in \mathbb{Z}\} = \{2^{j/2}\psi(2^j t - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for W_j. Then (by a change of variables) we get $V_{j+1} = V_j \oplus W_j$ for every integer j.

Since $W_j \perp V_j$ and $W_j \subseteq V_{j+1}$ for every integer j, $W_\ell \subseteq V_j$ if $\ell < j$. Hence $W_j \perp W_\ell$ if $\ell < j$. It follows that $W_{j_1} \perp W_{j_2}$ if $j_1 \neq j_2$. Thus $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is orthonormal.

Then we can say that, starting with any J,

$$V_J = W_{J-1} \oplus V_{J-1} = \cdots = W_{J-1} \oplus W_{J-2} \oplus \cdots \oplus W_{J-M} \oplus V_{J-M}. \tag{$S11$}$$

This is what is often wanted for applications. We can start with a signal $f(t)$ in V_J (actually, the projection of an "actual" signal onto V_J, or even an approximation) and write it as the sum of a "low-frequency" signal in V_{J-M} and a sum of signals with finer and finer detail, each one in a wavelet space with a different scale. Each of the wavelet spaces is perpendicular to the other wavelet spaces, and to V_{J-M}.

We would like to be sure that this works fully. We would like to be able to prove that the family of the ψ_{jk} forms an orthonormal basis for $L^2(\mathbb{R})$.

$(S11.4)$ **Theorem:** *For every* $f \in L^2$,

$$f(t) = \sum_j \sum_k \langle f,\, \psi_{jk}\rangle \psi_{jk}(t), \tag{$S11.5$}$$

an L^2 *series, where* $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$.

To prove this we recall that all we have to do is show that the only function (i.e., "vector") in L^2 that is orthogonal to every one of the ψ_{jk} is the zero function.

Proof: We know that the union of all the spaces V_j is dense in L^2. We can build on this information by showing that if f is orthogonal to every one of the ψ_{jk} then f is orthogonal to every one of the spaces V_j, hence orthogonal to their union. We can work one j at a time for the spaces V_j, so let's work with V_N. We are given that $f \perp \psi_{jk}$ for all j and k. This means that $f \perp W_j$ for all j. Let's let P_N denote the orthonormal projection on V_N. Then by $(S11)$,

$$P_N f = \sum_{j=1}^{M} \sum_{k} \langle P_N f,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t) + P_{N-M} f. \tag{$S12$}$$

Since $N > N-j$ in this range, we have $\langle P_N f,\, \psi_{N-j,k}\rangle = \langle f,\, P_N \psi_{N-j,k}\rangle = \langle f, \psi_{N-j,k}\rangle = 0$. Hence $P_N f = P_{N-M} f$ for all $M > 0$. Thus $P_N f \in V_j$ for all $j \le N$. But since the spaces are nested, $P_N f \in V_j$ for all $j > N$ as well. Hence $P_N f \in V_j$ for all j. We know that the intersection of all the spaces V_j is the zero subsapce of L^2. Thus $P_N f = 0$ for all N. But $P_N f = 0$ means that $f \perp V_N$. Since N was arbitrary, $f \perp V_N$ for all N, so $f \perp \bigcup_j V_j$, hence $f = 0$. **This completes the proof that the wavelets ψ_{jk} form an orthonormal basis for** L^2. Finally, we have shown that ψ *is* a wavelet.

Remarks

The equations $(S11.5)$ and $(S12)$ carry with them the idea that $P_N f \to f$ in L^2 as $N \to +\infty$, and the idea that $P_N f \to 0$ in L^2 as $N \to -\infty$. These ideas are correct.

§SW7 Introduction to applications of wavelets

Analysis of a signal

Suppose we start with a signal $f(t) \in L^2$ that has compact support. Usually we assume that $f(t)$ has as much smoothness as we need. But sometimes we have to deal with jump discontinuities. At any rate, we pick a finest scale of detail, N, and a coarsest one, $N-M$. We might sample the signal at lots of points, and make the assumption that $f(t_k) \approx 2^{N/2}\langle f,\, \varphi_{Nk}\rangle$, where the points t_k are pretty close to $k/2^{-N}$, and the k are chosen so that the $k/2^{-N}$ are in a supporting interval for f. The factor $2^{N/2}$ is there for normalizing – the integal of $2^{N/2}\varphi_{Nk}$ is one. If f were a constant, say K, on the support of φ_{Nk}, then $2^{N/2}\langle f,\, \varphi_{Nk}\rangle = K$. In other words, we think of

$$P_N f(t) \approx f_S(t) := \sum_{k\in S} 2^{-N/2} f(t_k)\varphi_{Nk}(t)$$

where the sum is taken over those k in a finite set S that corresponds to a support interval for f. Then we use $(S12)$, and say

$$P_N f(t) = \sum_{j=1}^{M}\sum_{k} \langle P_N f,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t) + P_{N-M} f(t) \approx f_S(t) = \sum_{j=1}^{M}\sum_{k} \langle f_S,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t) + P_{N-M} f_S(t).$$

Each sum $Q_{N-j} f_S(t) := \sum_k \langle f_S,\, \psi_{N-j,k}\rangle \psi_{N-j,k}(t)$ is finite and its coefficients $\langle f_S,\, \psi_{N-j,k}\rangle$ can be "computed exactly." Each f_{Sj} shows the detail that is present at the scale 2^{N-j}. Likewise, the term $P_{N-M} f_S(t)$ is a finite sum whose coefficients, $\langle f_S,\, \varphi_{N-M,k}\rangle$, can be "computed exactly." It shows the coarsest-scale details.

In all of this, "computed exactly" means, "expressed in terms of the numbers h_n and the coefficients of f_S." The only trouble is that these numbers may not be known exactly, so we once again have to rely on approximations.

Synthesis of a signal

We may want to save an approximation of a signal. One thing that is often done is to save the largest few percent of the coefficients, along with which wavelets or scaling functions they go with. Then only those coefficients and wavelets and scaling functions that are relevant are used to reconstruct, or synthesize, the signal (approximation).

Wavelets for $L^2(\mathbb{R}^n)$

Given a one-dimensional scaling function $\varphi(t)$ and its associated wavelet $\psi(t)$ we can define

$$\varphi(t_1,\, t_2,\, \dots,\, t_n) := \prod_{k=1}^{n} \varphi(t_k), \quad \text{and} \quad \psi^S(t_1,\, t_2,\, \dots,\, t_n) := \prod_{k\in S} \psi(t_k) \prod_{k\notin S} \varphi(t_k),$$

where S is a non-empty subset of $\{1, 2, \ldots, n\}$. This gives us one scaling function and $2^n - 1$ different "basic wavelets." Using this approach, it is a straightforward matter to redefine MRA and the wavelet subspaces, a family for each S. It can then be shown that the family $\{\psi_{jk}^{S} : j \in \mathbb{Z},\ k \in \mathbb{Z}^k,\ \emptyset \neq S \subseteq \{1, 2, \ldots, n\}\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Synthesis of other kinds of functions

Although wavelets "fit" into the context of Hilbert spaces, $L^2(\mathbb{R}^n)$, it has been found that many other spaces of functions defined on $\mathbb{R}^n$ can be synthesized using wavelets. Chapters 5 and 6 in the book of Hernandez and Weiss are devoted to this topic, and there are many papers on this subject of ongoing research.

Some of the kinds of functions wavelets are used to synthesize are smooth functions. Thus smooth wavelets are needed. This is where the finite sets of h_n's come in; with more work, we can arrange to have wavelets that have compact support and continuous derivatives of order up to any integer $N > 0$. The work involves (in effect) linear equations involving the h_n's that arise by demanding that more and more derivatives of $m_o(\xi)$ are zero when $\xi = \pi$ (we already require that $m_o(\pi) = 0$).

It is interesting that infinitely differentiable wavelets with compact support cannot exist!

§SW8 Appendix

To be added later