3: For $x \in \mathbb{R}$, define $H(x) := 1$ if $0 < x < 1/2$, $H(x) := -1$ if $1/2 < x < 1$, and let $H(x) := 0$ for all other x.
(i) Find j, k so that $H_{jk} = H$.
(ii) Shetch the graphs of $H_{2,1}$, $H_{-2,1}$, and $H_{3,-1}$.
(iii) Verify that the functions $H_{j,k}(x) = 2^{j/2}H(2^j x - k)$ satisfy

$$\langle H_{jk},\, H_{j'k'}\rangle = \begin{cases} 1 & \text{if } j' = j \text{ and } k' = k \\ 0 & \text{if } j' \neq j \text{ or } k' \neq k. \end{cases}$$

We say the H_{jk} *form an orthonormal system of functions in* $L^2(\mathbb{R})$ (these are the Haar functions; H is the Haar wavelet).

Solution for (iii) (Very expository!)

If $H_{jk}(t)H_{j'k'}(t) = 0$ a.e. (**a**lmost **e**verywhere), then

$$\langle H_{jk},\, H_{j'k'}\rangle = \int_{-\infty}^{\infty} H_{jk}(t)H_{j'k'}(t)\, dt = 0.$$

This is the easy case. When are we in the easy case? To answer this we figure out the "supporting interval" $I_{jk} = [a_{jk},\, b_{jk}]$ that contains all the points t and limit points of t's where $H_{jk}(t) \neq 0$. Since $H_{jk}(t) = 2^{j/2}H(2^j t - k)$, and $H(t)$ has the supporting interval $I = [0,\, 1] = I_{00}$, we get that $H_{jk}(t) = 2^{j/2}H(2^j t - k) = 0$ "for sure" unless $0 \leq 2^j t - k \leq 1$. Solving this inequality for t gives

$$I_{jk} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right].$$

If two of these supporting intervals do not overlap, then $H_{jk}(t)H_{j'k'}(t) = 0$ a.e. We can now answer the question "When are we in the easy case?"

Answer: When $\dfrac{k+1}{2^j} \leq \dfrac{k'}{2^{j'}}$ or when $\dfrac{k'+1}{2^{j'}} \leq \dfrac{k}{2^j}$. That is, when the supporting intervals do not overlap.

The next issue then is to see what happens when two supporting intervals *do* overlap. By the way, I and J "overlap" when $I \cap J$ has positive length. Thus $[-1,\, 1]$ and $[0,\, 2]$ overlap, while $[-1,\, 0]$ and $[0,\, 1]$ do not overlap because their intersection is a single point, which has length zero.

There are two relevant kinds of overlapping. One of them is $I \subseteq J$. In this case, both endpoints of I are contained in J (this is one reason why we are using *closed* intervals). The other kind of overlap requires that one endpoint of each interval be contained in the *interior* of the other. For example, $[-1,\, 1]$ and $[0,\, 2]$ overlap in this way, with $-1 < 0 < 1$ and $0 < 1 < 2$. The interior of $[0, 2]$ is $(0, 2)$; the interior of $(0, 2)$ is $(0, 2)$ itself.

The way our functions are (carefully!) defined, this last kind of overlap cannot occur. But suppose that it could happen. It would then be true that

$$(*) \qquad \frac{k'}{2^{j'}} < \frac{k}{2^j} < \frac{k'+1}{2^{j'}} < \frac{k+1}{2^j},$$

or else the similar inequalities with the primes moved would be true. The easiest case is $j' = j$. Then the inequalities become equivalent to

$$k' < k < k'+1 < k+1.$$

This is impossible because all are integers. In the other case we can assume that $j' > j$ (the case $j' < j$ is treated similarly). Then $j' = j + d$, where d is a positive integer. Then, if we multiply $(*)$ through by $2^{j'}$, we have

$$k' < 2^d k < k'+1 < 2^d(k+1).$$

This is impossible too because we now have an integer strictly between two consecutive integers. We can re-phrase what we have shown:

Two supporting intervals I_{jk} *and* $I_{j'k'}$ *can overlap if and only if one of them is contained in the other.*

This can happen in two ways: the intervals are identical, or one is a proper subset of the other.

If the intervals are the same, then $j = j'$ and $k = k'$, so

$$\langle H_{jk},\, H_{j'k'}\rangle = \langle H_{jk},\, H_{jk}\rangle = \int_{-\infty}^{\infty} H_{jk}(t)^2\, dt = \int_{k/2^j}^{(k+1)/2^j} 2^j\, dt = 1.$$

As we have seen, if $j' = j$ and $k' \neq k$, the supporting intervals of H_{jk} and $H_{j'k'}$ do not overlap so

$$\langle H_{jk},\, H_{jk'}\rangle = \delta_{kk'}.$$

We can now turn to the case $j' \neq j$. Let's assume that $j' > j$. Then $I_{j'k'} \subseteq I_{jk}$ and the length of $I_{j'k'}$ is at most half that of I_{jk}. We now need to rule out the possibility that the *midpoint* of I_{jk} is in the interior of $I_{j'k'}$. This would mean that (similar to $(*)$)

$$(*2) \qquad \frac{k'}{2^{j'}} < \frac{k + (1/2)}{2^j} < \frac{k'+1}{2^{j'}},$$

which gives (on multiplying through by $2^{j'}$)

$$k' < 2^d k + 2^{d-1} < k' + 1.$$

Again this imopssible because no integer lies between two consecutive integers.

Thus, if $j \neq j'$,

$$\langle H_{jk},\, H_{j'k'}\rangle = 0.$$

This completes this expository solution.

4: In Problem **3,** j refers to "scale," and k refers to "position." Find the Haar coefficients of the "ramp" function $r(x)$ that is equal to x for $-1 < x < 1$ and equal to zero elsewhere. That is, find

$$c_{jk} := \langle r,\, H_{jk}\rangle = \int_{-\infty}^{\infty} r(x)\overline{H_{jk}(x)}\, dx,$$

where the complex conjugation can be ignored since each H_{jk} is real-valued. I am particularly interested in how you arrange this two-parameter family of coefficients. One thought you might bear in mind is to compare your arrangement to sheet music.

The problem: calculate

$$c_{jk} = \int_{-1}^{1} tH_{jk}(t)\, dt,$$

where j and k both run thru *all* integer values, and

$$H_{jk}(t) := 2^{j/2}H(2^j t - k); \quad H(t) = 1 \text{ for } 0 < t < 1/2 \text{ and } H(t) = -1 \text{ for } 1/2 < t < 1; \text{ otherwise } H(t) = 0.$$

It will be useful to have handy the range of values of t where $H_{jk}(t)$ can be non-zero.

$$H_{jk}(t) \text{ can only be non-zero when } \frac{k}{2^j} < t < \frac{k+1}{2^j}.$$

This gives us an easy criterion that guarantees that $c_{jk} = 0$. That is, $c_{jk} = 0$ if $\frac{k+1}{2^j} \leq -1$ or if $\frac{k}{2^j} \geq 1$. This simplifies to

$$(1) \qquad c_{jk} = 0 \text{ if } k \leq -2^j - 1 \text{ or if } k \geq 2^j.$$

We can do this work in three "pieces:" $j = 0,\ j < 0,\ j > 0.$

Case: $j = 0$. The only viable values of k are $k = 0$ and $k = -1$, by (1). Then

$$c_{00} = \int_{-1}^{1} tH(t)\,dt = \frac{t^2}{2}\bigg|_0^{1/2} - \frac{t^2}{2}\bigg|_{1/2}^{1} = -\frac{1}{4}\,.$$

Next (and I will use "literal substitutions"),

$$c_{0,-1} = \int_{-1}^{1} tH(t+1)\,dt = \int_0^2 (t-1)H(t)\,dt = \int_0^1 (t-1)H(t)\,dt = \int_0^1 tH(t)\,dt = c_{00} = -\frac{1}{4}\,.$$

This phenomenom will recur: if $c_{jk} \neq 0$ and $c_{jk'} \neq 0$ then $c_{jk'} = c_{jk}$.

Case: $j < 0$. The only viable values of k are $k = 0$ and $k = -1$, by (1). Then (using that $H(t) = 0$ if $t < 0$ and using that $2^j \leq 1/2$)

$$c_{j0} = \int_0^1 t2^{j/2}H(2^j t)\,dt = \int_0^{2^j} \frac{t}{2^j}2^{j/2}H(t)\frac{dt}{2^j} = 2^{j/2}\int_0^{2^j} t\frac{dt}{2^{2j}} = 2^{j/2}\frac{2^{2j}}{2^{2j+1}} = 2^{j/2-1}.$$

Next,

$$c_{j,-1} = \int_{-1}^{1} t2^{j/2}H(2^j t+1)\,dt = \int_{-2^j}^{2^j} t2^{j/2}H(t+1)\frac{dt}{2^j} = -2^{j/2}\int_{-2^j}^{0} t\frac{dt}{2^j} = 2^{j/2}\frac{2^{2j}}{2^{j+1}} = 2^{j/2-1} = c_{j0}.$$

Case: $j > 0$. By (1), $c_{jk} = 0$ unless $-2^j - 1 < k < 2^j$. Suppose this is so. Then

$$c_{jk} = \int_{-1}^{1} t2^{j/2}H(2^j t - k)\,dt = 2^{j/2-2j}\int_{-2^j}^{2^j} tH(t-k)\,dt = 2^{j/2-2j}\int_{-2^j-k}^{2^j-k}(t+k)H(t)\,dt.$$

The condition on k can be rewritten $-2^j \leq k < 2^j$, so that

$$-2^j - k \leq 0 \;\text{ and }\; 0 < 2^j - k, \quad\text{or}\quad -2^j - k \leq 0 \;\text{ and }\; 1 \leq 2^j - k,$$

so that the region of integration contains the range where $H(t)$ is non-zero. Thus we can write

$$c_{jk} = 2^{j/2-2j}\int_0^1 (t+k)H(t)\,dt = 2^{j/2-2j}\int_0^1 tH(t)\,dt = 2^{j/2-2j}(-1/4) = -2^{-3j/2-2}$$

since $H(t)$, integrated against a constant, yields a zero integral.

Summary

$$c_{00} = -\frac{1}{4} = c_{0,-1};$$

$$c_{j0} = 2^{j/2-1} = c_{j,-1}, \quad \text{for all} \quad j < 0;$$

$$c_{jk} = -2^{-3j/2-2} \quad \text{for} \quad -2^j \le k < 2^j, \quad \text{for all} \quad j > 0;$$

$$\text{all other} \;\; c_{jk} = 0.$$

A check of norms (the Parseval formula)

The sum of the squares of the (absolute values) of the coefficients should agree with the integral of the square of the (absolute value) of the signal.

$$\int_{-1}^{1} t^2 dt = 2/3.$$

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk}^2 &= \sum_{j=0}^{0} \sum_{k=-1}^{0} \frac{1}{4^2} + \sum_{j=-\infty}^{-1} \sum_{k=-1}^{0} 2^{j-2} + \sum_{j=1}^{\infty} \sum_{k=-2^j}^{2^j-1} 2^{-3j-4} \\
&= \frac{1}{8} + \sum_{j=-\infty}^{-1} 2^{j-1} + \sum_{j=1}^{\infty} 2^{j+1} 2^{-3j-4} \\
&= \frac{1}{8} + \sum_{j=1}^{\infty} 2^{-j-1} + \sum_{j=1}^{\infty} 2^{-2j-3} \\
&= \frac{1}{8} + \frac{1}{2} + \frac{1}{8} \sum_{j=1}^{\infty} 4^{-j} \\
&= \frac{5}{8} + \frac{1}{8}\frac{1}{4}\frac{1}{1-(1/4)} = \frac{5}{8} + \frac{1}{8}\frac{1}{4}\frac{4}{3} = \frac{15+1}{24} = 2/3.
\end{aligned}$$

Remark Finding the coefficients completes the “analysis” part of the work. The “synthesis ” part is to verify that

$$f(t) := t[-1 < t < 1] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} H_{jk}(t)$$

in an appropriate sense. One point to make is that the double summation is NOT to be taken too literally! We can try to approximate $f(t)$ by looking at

$$\sum_{(j,\,k)\in\mathcal{F}} c_{jk} H_{jk}(t),$$

which means to sum over a set $\mathcal{F}$ of pairs $(j,\,k)$ of indices.