2(b): Suppose (for this part) $\int f(t)\,dt \neq 0$. Explain why it is impossible for $c_{jk} = 0$ to be true for all j sufficiently negative. The question behind this: what is the rôle of the very-low-frequency coefficients?

Our assumption that f is a *linear combination* of integer translates of the box function tells us that $f(t) = 0$ if $t < M$ or if $t > N$, where M and N are integers and $M < N$. The "worst case" is when $M < 0 < N$, so we will deal with that one. Then $-2^{J+1} < M \leq -2^J$ and $2^H \leq N < 2^{H+1}$, where J and H are non-negative integers.

Therefore $H_{-J-2,-1}(t) = -2^{-(J+2)/2}$ on $(M, 0) \subseteq (-2^{J+1}, 0)$ and $H_{-H-2,0}(t) = 2^{-(H+2)/2}$ on $(0, N)$. Since $\int f(t)\,dt \neq 0$, it has to be true that $\int_M^0 f(t)\,dt \neq 0$ or that $\int_0^N f(t)\,dt \neq 0$ and both may be true. Let us suppose that $\int_0^N f(t)\,dt \neq 0$ (the other case is treated similarly). Then

$$\langle f,\, H_{-J-2,-1}\rangle = \int_{-2^{J+2}}^0 f(t)H_{-J-2,-1}(t)\,dt = \int_M^0 f(t)H_{-J-2,-1}(t)\,dt = \int_M^0 f(t)(-2^{-(J+2)/2})\,dt \neq 0.$$

If $J' \geq J$ we thus have

$$\langle f,\, H_{-J-2,-1}\rangle = \int_M^0 f(t)(-2^{-(J+2)/2})\,dt = -2^{-(J+2)/2} \cdot \text{constant} \neq 0,$$

where "constant" stands for $\int_M^0 f(t)\,dt \neq 0$.

You were not asked to answer the "question behind all this." However, the answer is that the sum of all those coefficients we just looked at gives us a (negative) power of 2 times $\int_M^0 f(t)\,dt$, so the low-frequency coefficients are related to the integral of the signal.

In fact, this phenomenon happens even when the integral of f is zero, because it really happens precisely when the integral on either side of zero is non-zero. So, as in the example of the roof function, the integral was zero because the non-zero integrals on opposite sides of zero cancelled.

3: For $x \in \mathbb{R}$, define $H(x) := 1$ if $0 < x < 1/2$, $H(x) := -1$ if $1/2 < x < 1$, and let $H(x) := 0$ for all other x. We defined $H_{jk}(t) = 2^{j/2}H(2^j t - k)$.

The question to answer was:

Given j and k, which V_m does $H_{jk}(t)$ belong to?

Derivation: We can write $H(t) = B(2t) - B(2t-1)$, namely as a box of height 1 and width $1/2$ starting at $t = 0$, *minus* a box of height 1 and width $1/2$ starting at $t = 1/2$, where the first one left off. Therefore, $H(t) \in V_1$, and $H(t) \notin V_0$, because functions in V_0 have to be constant between consecutive integers.

Then,

$$H_{jk}(t) = 2^{j/2}H(2^j t - k) = 2^{j/2}\big(B\big(2(2^j t - k)\big) - B\big(2(2^j t - k) - 1\big)\big) = 2^{j/2}\big(B(2^{j+1}t - 2k) - B\big(2^{j+1}t - 2k - 1\big)\big).$$

After "cleanup," this reads

$$H_{jk}(t) = 2^{j/2}B(2^{j+1}t - 2k) - 2^{j/2}B(2^{j+1}t - 2k - 1)\big),$$

so $H_{jk}(t)$ is a box of height $2^{j/2}$ and width $1/2^{j+1}$ starting at $t = \dfrac{2k}{2^{j+1}}$, *minus* a box of height $2^{j/2}$ and width $1/2^{j+1}$ starting at $t = \dfrac{2k+1}{2^{j+1}}$, where the first one left off. Since boxes of width $1/2^{j+1}$ starting at numbers of the form $t = \dfrac{n}{2^{j+1}}$ are the "building blocks" for V_{j+1} and since boxes of width $1/2^{j+1}$ starting at numbers of the form $t = \dfrac{n}{2^{j+1}}$ are *not* in V_m for $m < j+1$, we have
Answer: $H_{jk} \in V_m$ if and only if $m > j$ (since $V_m \subseteq V_{j+1}$ if $m > j+1$).