1: Find the element of V_0 that is closest to $H_{00} + H_{11}$. Is there a simple "recipe" for solving this sort of minimization problem? If so, what is it?

The solution that I had in mind (there are many others!):

Let $f(t) := H_{00}(t) + H_{11}(t)$ By the Projection Theorem, we know that $P_{V_0} f$ is the unique element of V_0 such that $f - P_{V_0} f \perp V_0$. Then we write $P_{V_0} f(t) = \sum d_m B(t-m)$, where the numbers d_m are unknowns and where $B(t)$ is the box function equal to 1 in $(0, 1)$ and equal to zero elsewhere.

We know that every element of V_0 is expressible as a sum of coefficients times integer translates of the box function, so we know that (extra steps included)

$$\begin{aligned}
0 &= \langle f - P_{V_0} f(t),\, B(t-n)\rangle_{dt} \\
&= \langle f - \sum d_m B(t-m),\, B(t-n)\rangle_{dt} \\
&= \langle f,\, B(t-n)\rangle_{dt} - \langle \sum d_m B(t-m),\, B(t-n)\rangle_{dt} \\
&= \langle f,\, B(t-n)\rangle_{dt} - \sum d_m \langle B(t-m),\, B(t-n)\rangle_{dt} \\
&= \langle f,\, B(t-n)\rangle_{dt} - \sum d_m \delta_{mn} \\
&= \langle f,\, B(t-n)\rangle_{dt} - d_n,
\end{aligned}$$

so $d_n = \langle f,\, B(t-n)\rangle_{dt}$. Hence

$$P_{V_0} f(t) = \sum_n \langle f,\, B(s-n)\rangle_{ds} B(t-n).$$

This is **partial answer:** the simple "recipe."

In our case, since both of H_{00} and H_{11} are zero outside $(0, 1)$, $d_n = \langle f,\, B(s-n)\rangle_{ds} = 0$ unless $n = 0$.

But then, $B(t)$ is a constant where $H_{00}(t)$ and $H_{11}(t)$ are non-zero, and each of them has integral zero, so $d_0 = 0$.

Hence **partial answer:** $P_{V_0} f(t) = 0$ is the element of V_0 that is closest to $H_{00} + H_{11}$.

2: Calculate the orthogonal projection of $H_{00} + H_{11}$ onto V_{-1}. Is there a simple formula for doing this? If so, what is it?

In problem 1, we saw that the simple formula, which depends on the subspace, is **partial answer:**

$$P_{V_{-1}} f(t) = \sum_n \langle f,\, 2^{-1/2} B(2^{-1}s - n)\rangle_{ds}\, 2^{-1/2} B(2^{-1}t - n).$$

However, since $V_{-1} \subseteq V_0$, and since $H_{00} + H_{11} \perp V_0$, we also have $H_{00} + H_{11} \perp V_{-1}$, which means that **partial answer:** $P_{V_{-1}}(H_{00} + H_{11}) = 0$ also.