**Math 5467, Spring 2000: HW 4: solutions of # 2 and # 4**

**2:** Verify that the space $V_0$ generated by the integer translates of the "triangle scaling function" $T(t)$ shown in figure 2.2(b) generates a multiresolution analysis, even though the translates of the triangle function are not orthogonal to each other. You may, if you wish, take it for granted that the closure of the span of $\{T(t-n) : n \in \mathbb{Z}\}$ consists of all series $\sum_{n\in\mathbb{Z}} a_n T(t-n)$, where $\sum |a_n|^2 < \infty$. The main point of this problem is for you to *construct* the other spaces $V_j$ and check that they satisfy the multiresolution analysis conditions.

*Solution:* We begin with $V_0$, the collection of all functions

$$f(t) := \sum c_n T(t-n), \quad \text{where} \quad \sum |c_n|^2 < \infty.$$

This is a closed subspace. According to Figure 2.2(b), $T(t) = \frac{1}{2}T(2t) + T(2t-1) + \frac{1}{2}T(2t-2)$. This means that $T(t) = t$ for $0 < t \le 1$, $T(t) = 2 - t$ for $1 < t < 2$, and $T(t) = 0$ for all other $t$.

We now *define* the spaces $V_j := \{f(2^j t) : f(t) \in V_0\}$. Since $V_0$ is closed, all the $V_j$, being stretchings or shrinkings of $V_0$, are closed. This way of defining the $V_j$ is what lets us show that $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$. So we suppose $f(t) \in V_j$. Then $f(t) = g(2^j t)$ for some $g(t) \in V_0$. Then $f(2t) = g(2 \cdot 2^j t) = g(2^{j+1}t) \in V_{j+1}$ by construction. Conversely, if $f(2t) \in V_{j+1}$ then $f(2t) = g(2^{j+1}t)$ for some $g(t) \in V_0$. But then (replacing $t$ by $t/2$), $f(t) = g(2^j t)$, so by construction $f(t) \in V_j$.

The equation

$$T(t) = \frac{1}{2}T(2t) + T(2t-1) + \frac{1}{2}T(2t-2)$$

is the Scaling Equation for this collection of spaces; it shows that $T(t) \in V_1$, so the closure of the span of the integer translates of $T(t)$, which is $V_0$, satisfies $V_0 \subseteq V_1$. Therefore if $F(t) \in V_j$, then $F(t) = f(2^j t)$, where $f(t) \in V_0$. Hence $f(t) \in V_1$. But then $f(t) = g(2t)$, where $g(t) \in V_0$. Thus

$$F(t) = f(2^j t) = g(2 \cdot 2^j t) = g(2^{j+1}t), \quad \text{where } g(t) \in V_0. \text{ Hence } F(t) \in V_{j+1}.$$

Thus the spaces $V_j$ are nested.

We have checked conditions (i), (ii) and (iv) for a MRA.

To check (iii) we have two things to do. First, to show: $\bigcap_j V_j = \{0\}$. First we recall that all our functions in a $V_j$ are continuous. Moreover, the graph of $f(t)$ can hve corners only at $t$ of the form $t = m/2^j$.

Suppose $f \in \bigcap_j V_j = \{0\}$. Then $f \in V_0$, so that $f(t) = \sum c_n T(t-n)$, where $\sum |c_n|^2 < \infty$, then for each integer $m$, $f(m+1) = \sum c_n T(m+1-n) = c_m$ because the only *integer* value of $t$ at which $T(t) \neq 0$ is $t = 1$. And $T(1) = 1$. Hence in the sum, when $n = m$, $m + 1 - n = 1$, and then $T(m+1-n) = 1$. For all other $n$, $T(m+1-n) = 0$.

Among all the numbers $|c_n|$ there must be at least one that is maximal. Let's pick the *largest* $n$ such that $|c_n|$ is maximal, and call it $N$. Then for all $t$, $|f(t)| \le |c_N| = |f(N+1)|$, and $|f(t)| < |c_N|$ if $t > N+1$. That is, the graph of $f(t)$ has a *corner* at $t = N+1$. Now $f \in V_{-1}$ also, and functions there can have corners only at *even* integers. Thus $N+1$ is even. Continuing, we see that $N+1$ must be divisible by evey positive power of 2. Thus $N+1 = 0$. But then where is the next corner? There must be one, if $f$ is not the zero function, because $|f(0)|$ is maximal, and $|f(t)| < |f(0)|$ if $t > 0$. This gives a contradiction, because the corner must occur at a point that is an integer divisible by every positive power of 2. No positive integer has this property. Hence $f(t) \equiv 0$.

We also have to show that $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$. To do this we apply a "Fact," that the continuous functions with compact support are dense in $L^2(\mathbb{R})$. Another fact we need is that such functions are *uniformly continuous,* meaning that for every given $\epsilon > 0$ there is a $\delta > 0$ that works for all $x$. That is, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. This can't be true for a function such as $f(x) = x^2$. The $\delta > 0$ depends on $x$, and is very small if $|x|$ is large.

We will actually (along the way) prove another "Fact:" piecewise linear continuous functions with compact support are dense in $L^2$. We will actually show that continuous piecewise linear functions with compact support that have dyadic rational "nodes" are dense. These are precisely the functions in $\bigcup_j V_j$.

We need an **Observation:** For every $N$, $\sum_n T(2^N t - n) \equiv 1$.

Now: we are given $f(t) \in L^2$, and given $\epsilon > 0$. We want to find $v(t) \in \bigcup_j V_j$ such that $\|f - v\| < \epsilon$.

Step 1: Find $g(t)$, a continuous function with compact support, such that $\|f - g\| < \epsilon/2$. For this problem it was given in class that this can be done. We can talk about *how* later, if you like. We next (if necessary) enlarge the supporting interval of $g$ so that its endpoints are dyadic rational numbers, say $m_1/2^K$ and $m_2/2^K$. Let's set $L := (m_2 - m_1 + 1)/2^K$. The $+1$ is there for later "sloppy" reasons.

Step 2: Approximate $g(t)$ *uniformly* by a function

$$(*) \qquad v(t) = \sum_n g((n+1)/2^N) T(2^N t - n).$$

I.e., we want to show that $|g(t) - v(t)| < \alpha$ for some $\alpha > 0$ that we will choose later to make sure that $\|g - v\| < \epsilon/2$.

We can do this by applying the fact that $g(t)$ is uniformly continuous. Thus we take the $\alpha > 0$ and find $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \alpha$. Then we find $N$ so large that $N \geq K$ and $1/2^N < \delta$. Now we define $v(t)$ by the formula in $(*)$. Using the Observation we have

$$g(t) - v(t) = \sum_n g(t) T(2^N t - n) - \sum_n g((n+1)/2^N) T(2^N t - n) = \sum_n (g(t) - g((n+1)/2^N)) T(2^N t - n).$$

The individual term $(g(t) - g((n+1)/2^N)) T(2^N t - n)$ is zero unless $0 < 2^N t - n < 2$. That is, unless $n/2^N < t < (n+2)/2^N$. This in turn means that $|t - (n+1)/2^N| < 1/2^N$ where the term is non-zero.

But then $|g(t) - g((n+1)/2^N)| < \alpha$. Hence

$$|g(t) - v(t)| < \sum_n \alpha T(2^N t - n) = \alpha.$$

To figure out what $\alpha$ has to be, we estimate $\|g - v\|^2$.

$$\int |g(t) - v(t)|^2 dt < \alpha^2 \int_{m_1/2^K}^{(m_2+1)/2^K} dt = \alpha^2 L.$$

Now we see that we should have defined $\alpha = \epsilon/2\sqrt{L}$. Then all the steps above work, and we get the desired result.

**4:** Calculate the Fourier transform $\widehat{G}(\xi)$ of the "Gaussian" function $G(x) := \dfrac{e^{-x^2}}{\sqrt{\pi}}$. The following outline is suggested. There are other methods of solution, including the use of a contour integral.
1) Write $\varphi(\xi) := \int e^{-x^2} e^{-ix\xi} dx$. The limits of integration are minus and plus infinity. We showed in class that $\varphi(0) = \sqrt{\pi}$. Calculate $\varphi'(\xi)$ by differentiating under the integral sign.
2) Integrate by parts and simplify. You will obtain an ordinary differential equation (ODE) for $\varphi$.
3) Solve the differential equation, and put the $\sqrt{\pi}$ back in place. One way uses a "first chapter" ODE method, another uses an integrating factor. See me if you need help with ODE; this part of ODE is so easy it's fun.

*Solution:* $\varphi'(\xi) = \int e^{-x^2}(-ix)e^{-ix\xi} dx = \frac{i}{2} \int (-2xe^{-x^2}) e^{-ix\xi} dx$. This is $0$ if $\xi = 0$ (odd function). If $\xi \neq 0$, integration by parts gives $\varphi'(\xi) = -\frac{i}{2} \int e^{-x^2}(-i\xi) e^{-ix\xi} dx = \frac{-\xi}{2} \int e^{-x^2} e^{-ix\xi} dx = -\xi\varphi(\xi)/2$. This gives us the linear ODE $y' + xy/2 = 0$, $y(0) = \sqrt{\pi}$. We can say $dy/y = -x/2$, so $\log y = -x^2/4$. Hence we try $\varphi(\xi) = \sqrt{\pi} e^{-\xi^2/4}$. This works. Uniqueness applies because $-xy/2$ is locally Lipschitz in $y$. We have shown that

$$\widehat{G}(\xi) = e^{-\xi^2/4}.$$