

3: The *Shannon scaling function* is $\varphi(t) = \text{sinc}(t) = \frac{\sin \pi t}{\pi t}$. Find $\widehat{\varphi}(\xi)$ and the coefficients $h(n)$ for its dilation equation. Find the *formula* (not a series) for the Shannon wavelet.

A solution The idea here is **not** to try finding the Fourier transform directly, but to use something we already know. We know that $\widehat{B}(\xi) = e^{-i\xi/2} \frac{\sin \xi/2}{\xi/2}$ and we know that $e^{-i\xi/2}$ signifies a translation to the right by $1/2$ so

$$\widehat{B}(\xi) = (B(t + (1/2)) - (1/2))\widehat{(\xi)} = e^{-i\xi/2}(B(t + (1/2)))\widehat{(\xi)} = e^{-i\xi/2} \frac{\sin \xi/2}{\xi/2}$$

and therefore $(B(t + (1/2)))\widehat{(\xi)} = \frac{\sin \xi/2}{\xi/2}$. If we replace ξ here by $2\pi\xi$ we get

$$\frac{\sin \pi\xi}{\pi\xi} = (B(t + (1/2)))\widehat{(2\pi\xi)} = \frac{1}{2\pi} 2\pi(B(t + (1/2)))\widehat{(2\pi\xi)} = \frac{1}{2\pi} B\left(\frac{t}{2\pi} + (1/2)\right)\widehat{(\xi)} = \frac{1}{2\pi} B\left(\frac{t + \pi}{2\pi}\right)\widehat{(\xi)}.$$

Now $B\left(\frac{t + \pi}{2\pi}\right)$ is a unit height box function that is one if and only if $0 < \frac{t + \pi}{2\pi} < 1$, or $0 < t + \pi < 2\pi$, or $-\pi < t < \pi$. Thus $\frac{\sin \pi\xi}{\pi\xi}$ is the Fourier transform of $1/2\pi$ time the box function of width 2π centered at 0. Let's just write

$$\frac{\sin \pi\xi}{\pi\xi} = \frac{1}{2\pi} [-\pi, \pi]\widehat{(\xi)}.$$

Next we will use the Fourier Inversion formula:

$$\left(\frac{\sin \pi\xi}{\pi\xi}\right)\widehat{(t)} = \frac{1}{2\pi} [-\pi, \pi]\widehat{(t)} = \frac{1}{2\pi} 2\pi [-\pi, \pi](-t) = [-\pi, \pi](t) = [-\pi \leq t \leq \pi],$$

that is, $\widehat{\varphi}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)\widehat{(t)}$ is the box function of height 1, width 2π , center 0. The Fourier version of the Scaling Equation is $\widehat{\varphi}(\xi) = m_o(\xi/2)\widehat{\varphi}(\xi/2)$. In our case it becomes

$$[-\pi \leq \xi \leq \pi] = m_o(\xi/2)[- \pi \leq \xi/2 \leq \pi] = m_o(\xi/2)[-2\pi \leq \xi \leq 2\pi].$$

In the interval $[-2\pi, 2\pi]$ we thus need $m_o(\xi/2) = [-\pi \leq \xi \leq \pi]$ which is the same as: in the interval $[-\pi, \pi]$,

$$m_o(\xi) = [-\pi/2 \leq \xi \leq \pi/2].$$

We know m_o is periodic of period 2π , and in $L^2(\mathbb{T})$, so $m_o(\xi) = \sum_{n \in \mathbb{Z}} c_n e^{in\xi}$, where $c_n = \frac{1}{2\pi} \int_{\mathbb{T}} m_o(\xi) e^{-in\xi} d\xi$.

But we have written this differently: $m_o(\xi) = \sum_{n \in \mathbb{Z}} \frac{h(n)}{\sqrt{2}} e^{-in\xi}$. Thus we have to replace c_n by c_{-n} and include the $\sqrt{2}$ in the denominator. I.e., (and because we can integrate over any interval of length 2π),

$$c_{-n} = \frac{h(n)}{\sqrt{2}} = \frac{1}{2\pi} \int_0^{2\pi} e^{in\xi} m_o(\xi) d\xi = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{in\xi} d\xi = \frac{e^{in\xi}}{2\pi in} \Big|_{-\pi/2}^{\pi/2} = \frac{\sin(n\pi/2)}{2(n\pi/2)}.$$

When n is even, $\sin(n\pi/2) = 0$, and when n is odd, $\sin(n\pi/2) = (-1)^{(n-1)/2}$. Hence

$$\frac{h(n)}{\sqrt{2}} = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \frac{(-1)^{(n-1)/2}}{n}, & \text{if } n \text{ is odd.} \end{cases}$$

Next we have to find a wavelet corresponding to the Shannon scaling function. We will use the Fourier transform definition:

$$\widehat{\psi}(\xi) := e^{i\xi/2} \overline{m_o(\xi/2 + \pi)} \widehat{\varphi}(\xi/2).$$

Thus $\widehat{\psi}(\xi) = e^{i\xi/2}[-\pi/2 \leq \xi/2 + \pi \leq \pi/2][-2\pi \leq \xi \leq 2\pi] = e^{i\xi/2}[-3\pi \leq \xi/2 \leq -\pi][-2\pi \leq \xi \leq 2\pi]$, or, simplifying, $\widehat{\psi}(\xi) = e^{i\xi/2}[-2\pi \leq \xi/2 \leq -\pi]$. This leads us again to use the Fourier Inversion formula.

$$\psi(t) = \frac{1}{2\pi} \widehat{\psi}(-t) = \frac{1}{2\pi} \int_{-2\pi}^{-\pi} e^{it\xi} e^{i\xi/2} dt = \frac{e^{-i\pi(t+1/2)} - e^{-i2\pi(t+1/2)}}{2\pi i(t+1/2)} = e^{-i3\pi(t+1/2)/2} \frac{\sin \pi(t+1/2)/2}{2\pi(t+1/2)/2}.$$

We notice that, using the formula for a wavelet that we used, we started with a real-valued scaling function and arrived at a complex-valued wavelet.

Remarks on 1, 2, 4.

Anent **1**: When we use conditions (h1), (h2) and (h3), we find that one of our three supposedly non-zero h_n must be zero! Thus, no such triples of h_n 's exist.

Anent **2**: Phi Fact 4 is the Fouriertransform version of this, the Scaling Equation. When we write the RHS as a sum of complex exponentials times $\widehat{\psi}(\xi/2)$, and interpret each one as the Fouriertransform of a translate of $2\psi(2t)$, use the appropriate Fourier Facts, we get the RHS in Problem 2.

Anent **4**: Here, I had in mind using the Theorem that the integer translates of an $L^2(\mathbb{R})$ function $g(t)$ form an o.n. set if and only if $\sum_k |\widehat{g}(\xi + 2\pi k)|^2 = 1$ a.e. This is made-to-order for the Fouriertransform of the Shannon scaling function! However, you can use the Plancherel Formula, and this works nicely too – though you do have to evaluate the integral of $e^{in\xi}$ over an interval of length 2π .