Math 5616H, Spring 2006 **Connected open sets in $\mathbb{R}^n$ are path connected**

Here is an important feature of the topology of $\mathbb{R}^n$: points in connected *open* sets can be "connected" to each other by a continuous broken-line path that lies in the connected open set. The standard name for continuous broken-line paths is "polygonal paths." The proof does not tell us *how* to construct the paths!

(1) **Definition:** *If $S \subseteq \mathbb{R}^n$ a polygonal path in S is a continuous function p defined on a nonempty compact interval $[a, b]$ that takes its values in S and there is a partition $\pi|[a, b]$ such that for $1 \le k \le n_\pi$,*

$$p(t) = \frac{t_k - t}{t_k - t_{k-1}} p(t_{k-1}) + \frac{t - t_{k-1}}{t_k - t_{k-1}} p(t_k) \quad \text{when} \quad t_{k-1} \le t \le t_k.$$

We say that p connects $p(a)$ and $p(b)$ in S.

(2) **A construction: joining paths end-to-end**

We notice that if p connects x_0 and x_1 is S and q connects x_1 and x_2 (both p and q being polygonal paths in S, with $p : [a, b] \to S$ and $q : [c, d] \to S$) we can create a polygonal path r in S that connects x_0 and x_2. We can define $r : [a, b + (d - c)] \to S$ by $r(t) := p(t)$ if $a \le t \le b$ and by $r(t) = q(c + (t - b))$ if $b \le t \le b + (d - c)$. At $t = b$, r is well-defined because $p(b) = x_1 = q(c) = q(c + (b - b))$. We also note that $r(t) \in S$ for all $t \in [a, b + (d - c)]$ and that $r(a) = x_0$ and $r(b + (d - c)) = q(c + ([b + (d - c)] - b)) = q(d) = x_2$. We can denote r by $p \oplus q$ to emphasize this construction.

(3) **Lemma:** *If $\Omega \subseteq \mathbb{R}^n$ is open, nonempty and connected then for every pair of distinct points x_0 and x_1 in Ω there exists a polygonal path $p : [0, 1] \to \Omega$ such that $p(0) = x_0$ and $p(1) = x_1$.*

Proof: Let $x_0 \in \Omega$. We define A to be the set of all points x in Ω such that there exists a polygonal path in Ω that connects x_0 and x. Since Ω is open and $x_0 \in \Omega$ there exists $\delta > 0$ such that $B_\delta(x_0) \subseteq \Omega$. For every $x \in B_\delta(x_0)$,

(4) $$p_{x_0x}(t) := (1 - t)x_0 + tx \quad \text{when} \quad 0 \le t \le 1$$

is a polygonal path in Ω that connects x_0 and x. Thus $A \neq \emptyset$. We note that $p_{x_0x_0}(t) \equiv x_0$ makes sense!

Next we claim that A is open. If $x \in A \subseteq \Omega$ then there exists $\eta > 0$ such that $B_\eta(x) \subseteq \Omega$. There is a polygonal path p in Ω that connects x_0 and x. For each $y \in B_\eta(x)$ the path $p \oplus p_{xy}$ is a polygonal path in Ω that connects x_0 and y. Hence $y \in A$, so A is open.

Next we claim that if $y \in \partial A$ then $y \notin \Omega$. Suppose not. Thus $y \in \partial A$ and $y \in \Omega$. Because $y \in \Omega$ there exists $\zeta > 0$ such that $B_\zeta(y) \subseteq \Omega$. Because $y \in \partial A$ there exists $z \in A \cap B_\zeta(y)$ so there exists a polygonal path s in Ω that connects x_0 and z. But then $s \oplus p_{yz}$ is a polygonal path in Ω that connects x_0 and y. This is a contradiction because A is open and hence contains none of its boundary points. Hence $y \notin \Omega$.

As our next step we define $B := \Omega \setminus A$. We will complete the proof by showing that, if B is *not* empty, A and B are separated sets, contradicting the given connectedness of Ω.

Thus we suppose that some $x_1 \in B$. Since $B \subseteq \Omega$ there exists $\gamma > 0$ such that $B_\gamma(x_1) \subseteq \Omega$. But every point $y \in B_\gamma(x_1)$ can be connected to x_1 by the polygonal path p_{yx_1}, so $y \notin A$. We have shown that B is open.

As we did for A we claim that if $y \in \partial B$ then $y \notin \Omega$. Otherwise $y \in A$ and some neighborhood $B_\epsilon(y) \subseteq \Omega$. Since $y \in \partial B$ there exists $z \in B_\epsilon(y) \cap B$. But then, if u is a polygonal path in Ω that connects x_0 and y, $u \oplus p_{yz}$ is a polygonal path in Ω that connects x_0 and z. This implies that $z \in A$, a contradiction since A and B are disjoint.

Our work has shown that (assuming $B \neq \emptyset$) A and B are nonempty sets whose union is Ω, that $\overline{A} \cap B = \emptyset$ and that $A \cap \overline{B} = \emptyset$. This contradicts the connectedness of Ω and completes the proof of the Lemma, since x_0 was an arbitrary point in Ω.