Math 5615H, Fall 2003 Nonempty perfect sets in $\mathbb{R}^k$ are uncountable 1 of 1

Introduction

Let P denote our nonempty perfect set. Then P must be infinite (why?). We will use contradiction, reached via a construction. The "construction" will be "by induction." The way such constructions are often done is "not quite right," but fortunately usually (accidentally) valid. We will use the Recursion Theorem to "do it right." You will not be tested on the Recursion Theorem; you may use it, if you do so correctly.

Our contradiction assumption is that P is countable. We will construct a nested sequence of nonempty compact sets that has empty intersection to get the contradiction. We can write $P = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$.

Observation

Suppose we know, about some $n \in \mathbb{Z}^+$, and some $r > 0$, that, with $V := B_r(x_n)$, $\overline{V}$ contains *no* x_j with $j < n$. We can then show that there exist $m > n$ and $s > 0$ such that, with $W := B_s(x_m)$, $\overline{W} \subseteq V$ and $\overline{W}$ contains *no* x_j with $j < m$.

Proof: We define $T := \{j \in \mathbb{N} : j > n \text{ and } x_j \in V\}$. Then T has a least element, which we call m. Since $m \in T$, $m > n$. We set $s := \min\{|x_n - x_m|, r - |x_n - x_m|\}/2$. Since $x_m \neq x_n$ and $x_m \in V$, $0 < |x_n - x_m| < r$, so $s > 0$. Now, $W \subseteq V$, and, because of the division by 2 in the definition of s, $\overline{W} \subseteq V$. Thus $\overline{W}$ certainly contains no x_j with $j \leq n$. Moreover, among all j with $j > n$, m is the least j such that $x_j \in W$. Thus $\overline{W}$ contains *no* x_j with $j < m$, and the proof is complete.

An example showing that there is a "starting" pair (n, r).

To begin our construction we choose $r_1 = 1$, $n_1 = 1$, and set $V_1 := B_{r_1}(x_{n_1})$. Then the conditions of the Observation are true, since there simply *are* no x_j with $j < 1$. Thus there exist $n_2 > n_1$, and $r_2 > 0$, that we obtain from the Observation, such that with $r_2 := \min\{|x_{n_1} - x_{n_2}|, 1 - |x_{n_1} - x_{n_2}|\}/2 > 0$ and $V_2 := B_{r_2}(x_{n_2})$, $\overline{V_2} \subseteq V_1$ and $\overline{V_2}$ contains *no* x_j with $j < n_2$. The conditions of the Observation are thus true for n_2 and r_2.

A paragraph that is not really needed this is the way this is often done...

To continue by induction, we suppose that we have positive integers n_ℓ and positive numbers r_ℓ, $1 \leq \ell \leq L$, such that $n_\ell < n_{\ell+1}$, for $1 \leq \ell < L$, and we have constructed V_ℓ, $1 \leq \ell \leq L$, with $V_\ell = B_{r_\ell}(x_{n_\ell})$, such that none of the sets $\overline{V_\ell}$ contains any x_j with $j < n_\ell$, and such that $\overline{V_{\ell+1}} \subseteq V_\ell$ for $1 \leq \ell < L$. We can now apply the Observation to x_{n_L} and r_L to find n_{L+1} and $r_{L+1} > 0$ such that, with $V_{L+1} := B_{r_{L+1}}(x_{n_{L+1}})$, $\overline{V_{L+1}} \subseteq V_L$ and $\overline{V_{L+1}}$ contains *no* x_j with $j < n_{L+1}$. The conditions of the Observation are thus true for n_{L+1} and r_{L+1}.

How we really justify definition by induction

In order to say that this process *defines* a sequence inductively, we need to apply one more major Theorem, that we will treat as an axiom (we could prove it but it would be a big digression). I learned this recently from Leonard Blackburn, a recent Minnesota PhD.

The Recursion Theorem *Let Y be a non-empty set, and suppose that $y_1 \in Y$. Suppose also that $H : Y \to Y$ is a function. Then there exists a unique sequence $R : \mathbb{Z}^+ \to Y$ such that $R_1 = y_1$ and such that for all $n \in \mathbb{Z}^+$, $R_{n+1} = H(R_n)$.*

We will apply the Recursion Theorem to $Y \subseteq \mathbb{Z}^+ \times \mathbb{R}^+$ ($\mathbb{R}^+$ denotes the set of positive real numbers), where $Y := \{(n, r) \in \mathbb{Z}^+ \times \mathbb{R}^+ : \overline{B_r(x_n)} \text{ contains no } x_j \text{ with } j < n\}$. Our function $H : Y \to Y$ will be given by $H(n, r) = (m, s)$, as found in the Observation, and our $y_1 := (1, 1)$. What we have to know is that H is defined on Y and takes values in Y. This is what the Observation does. This leaves open the possibility that Y is empty. We showed that $y_1 = (1, 1)$ is in Y, so Y is nonempty. We can now apply the Recursion Theorem and conclude that the associated neighborhoods V_ℓ exist for all $\ell \in \mathbb{Z}^+$. It might help to read the first (long!) sentence of the not-needed paragraph now, replacing "$< L$" and "$\leq L$" by $< \infty$.

Now we define $K_\ell := \overline{V_\ell} \cap P$. The K_ℓ are compact (why?), nested (which means what?) and nonempty (why?). Thus $\bigcap_{\ell=1}^{\infty} K_\ell$ is nonempty. But since each $K_\ell \subseteq P$, $\bigcap_{\ell=1}^{\infty} K_\ell \subseteq P$. However, for each ℓ, K_ℓ contains no x_j with $j < n_\ell$. We know that $n_\ell \geq \ell$ for all ℓ. It follows (how?) that no x_L is in $\bigcap_{\ell=1}^{\infty} K_\ell \subseteq P$, which is therefore empty. This is a contradiction. The proof is complete.