The exponential series

For $z \in \mathbb{C}$ the function $\exp(z)$ is defined by the power series

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

which converges absolutely, by the Ratio Test, for all $z \in \mathbb{C}$. As an application of the M-test, if $|z| \leq A$ the series of functions $z^n/n!$ converges uniformly, so $\exp(z)$ is continuous, as a function of z. We have shown that $\exp(z)$ is differentiable, in the complex sense, and that $\exp'(z) = \exp(z) =: e^z$, for all $z \in \mathbb{C}$.

Some miscellaneous properties of the exponential function

1: $\exp(\overline{z}) = \overline{\exp(z)}$. This follows from examining the series.

2: $e^0 = \exp(0) = 1$. This follows from examining the series.

3: For all $z \in \mathbb{C}$, $e^z = \exp(z) \neq 0$. We have $1 = \exp(0) = \exp(z)\exp(-z)$.

The trigonometric functions

The miscellaneous properties and the Law of Exponents tell us that, on the imaginary axis, the exponential has absolute value one. That is, its values lie on the unit circle. We define the functions $\cos x$ and $\sin x$ to be, respectively, the real and imaginary parts of $\exp(ix)$. Thus

Definition: $\cos x := \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x := \dfrac{e^{ix} - e^{-ix}}{2i}$.

Remark: We can use these definitions when x is complex as well. For the moment, we will only study the case of real x.

By the Chain Rule, or by the same sort of argument we used to show differentiability in the complex sense, we show (using the definitions) that

$$\frac{d}{dx}\cos x = -\sin x, \quad \text{and} \quad \frac{d}{dx}\sin x = \cos x.$$

We need to associate a point on the unit circle with each real x. Conversely, given a point z on the unit circle we want to know all real t such that $z = e^{it}$, if any.

Lemma: *There exists a positive number θ such that $e^{i\theta} = i$.*

Proof: To begin, let's examine $e^i = a + ib := \cos 1 + i \sin 1$. We have

$$e^i = \sum_{n=0}^{\infty} \frac{i^n}{n!} = \sum_{n=0}^{\infty} \left\{ \left(\frac{1}{(4n)!} - \frac{1}{(4n+2)!} \right) + i \left(\frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right) \right\}.$$

After simplification we get

$$e^i = \sum_{n=0}^{\infty} \left\{ \left(\frac{16n^2 + 12n + 1}{(4n+2)!} \right) + i \left(\frac{16n^2 + 20n + 5}{(4n+3)!} \right) \right\}.$$

Thus $a > 0$ and $b > 0$. The sum of the first two real parts is smaller than the first imaginary part:

$$\frac{0+0+1}{(2)!} + \frac{16+12+1}{(4+2)!} = \frac{1}{2} + \frac{29}{720} < \frac{1}{2} + \frac{1}{3} = \frac{5}{6},$$

the last quantity being the first imaginary part. After that, we have

$$\frac{16(n+1)^2 + 12(n+1) + 1}{(4(n+1)+2)!} \frac{(4n+3)!}{16n^2 + 20n + 5} = \frac{16n^2 + 44n + 29}{16n^2 + 20n + 5} \frac{1}{(4n+6)(4n+5)(4n+4)}.$$

By long division, $\dfrac{16n^2+44n+29}{4n+4} = 4n+7+\dfrac{1}{4n+4}$, so the ratio of "real part $n+1$ over imaginary part n," when $n \geq 1$, is

$$\frac{4n+7+1}{16n^2+20n+5}\frac{1}{(4n+6)(4n+5)} < \frac{4n+8}{20n+20}\frac{1}{(4n+6)(4n+5)} < 1.$$

Hence $0 < a < b$. Then $e^{2i} = (a+ib)^2 = a^2 - b^2 + 2iab$, so $\cos 2 < 0 < \cos 1$. Hence there exists $\theta > 0$ such that $\cos\theta = 0$. Since the set of all such θ is closed (because $\cos 0 = 1$), we choose the least such θ, and call it $\pi/2$.

Let us show that $1 < \pi/2 < 2$. The second inequality is immediate. To show the first, we note that, if $0 < x \leq 1$, then

$$e^{ix} = \sum_{n=0}^{\infty}\left\{\frac{x^{4n}}{(4n)!}\left(1-\frac{x^2}{(4n+2)(4n+1)}\right) + i\frac{x^{4n+1}}{(4n+1)!}\left(1-\frac{x^2}{(4n+3)(4n+2)}\right)\right\}.$$

Each real part and each imaginary part is positive, so $\cos x$ and $\sin x$ are both positive on $[0,\,1]$. Thus $\pi/2 > 1$. Moreover, $\cos x > 0$ on $[0,\,\pi/2)$, so $\sin(\pi/2) > 0$. Hence $\sin(\pi/2) = 1$ because $\sin^2(\pi/2) = 1$.

We have shown that $e^{i\pi/2} = i$.

Corollary: *Let $\pi/2$ denote the smallest positive number θ such that $\cos\theta = 0$. Then*

$$e^{i\pi/2} = i,\ e^{i\pi} = -1,\ e^{i3\pi/2} = -i,\ e^{2\pi i} = 1.$$

Moreover, $\cos x$ decreases strictly, and $\sin x$ increases strictly on $[0,\,\pi/2]$, and each has image $[0,\,1]$ there.

Proof includes use of the Intermediate Value Theorem and **5.11**. As an application of the foregoing we can now prove the following Theorem.

Theorem 1: *The function $U(\theta) := e^{i\theta}$ is periodic of period 2π, and maps the interval $[0,\,2\pi)$ onto the unit circle in the complex plane in a one-to-one manner.*

Proof: We know from the Corollary that U is one-to-one on $[0,\,\pi/2]$. Let us show that the image of $[0,\,\pi/2]$ with respect to U is the portion of the unit circle in the closed first quadrant. We know already that $U(0) = 1$ and that $U(\pi/2) = i$. Suppose that $z = x+iy$ and that x and y are both positive, with $x^2+y^2 = 1$. Then there exists $\theta \in (0,\,\pi/2)$ such that $\cos\theta = x$. Then $0 < \sin\theta = y$ since $\sin^2\theta = 1-x^2 = y^2$ and $y > 0$.

Next we consider the behavior of U on $[\pi/2,\,\pi]$. If $\pi/2 < \theta < \pi$, we set $\eta := \pi - \theta$. Then

$$e^{i\theta} = e^{i\pi-\eta} = -e^{-i\eta} = -\overline{e^{i\eta}} = -\overline{(\cos\eta + i\sin\eta)} = -(\cos\eta - i\sin\eta) = -\cos\eta + i\sin\eta$$

which shows that the quarter-circle in the closed first quadrant is mapped onto the quarter-circle in the closed second quadrant in a one-to-one manner. We make use of the invertible linear mapping whose matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and the fact that the composition of one-to-one and onto mappings is one-to-one and onto. In particular, to be sure that the map is onto the quarter-circle in the closed second quadrant, we can find the inverse image of a point thereon, which will be a point on the quarter-circle in the closed first quadrant, hence find an appropriate $\eta \in (0,\,\pi/2)$, then set $\theta = \pi - \eta$.

This shows that the function U maps the interval $[0,\,\pi]$ onto the part of the unit circle in the closed upper half plane.

Finally, the linear mapping given by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ maps the part of the unit circle in the open upper half plane onto the part of the unit circle in the open lower half plane in a one-to-one manner. It follows that the mapping $U : [0,\,2\pi) \to \mathbb{C}$ is a one-to-one and onto mapping of that interval to the unit circle.

More on the exponential function

The exponential function is periodic of period $2\pi i$. Thus its behavior can be described by restricting our attention to a certain strip.

Theorem 2: *Let S denote the strip $\{z \in \mathbb{C} : z = x + iy$ and $0 \le y < 2\pi\}$. Then the mapping $\exp : S \to \mathbb{C}$ given by $w = \exp(z)$ is one-to-one, and takes on all non-zero values of w.*

Proof: To prepare for the proof we need to discuss the behavior of the exponential function on the real axis. It follows from examination of the series that e^x is real when x is real. The same examination yields the result that $e^x \ge 1$ for $x \ge 0$. Then the identity $1 = e^x e^{-x}$ shows that $0 < e^x \le 1$ for $x \le 0$. Then the derivative of e^x being e^x shows that e^x is strictly increasing on $\mathbb{R}$. Since $e^1 > 1$, the numbers e^n tend to $+\infty$ as $n \to +\infty$, and tend to 0 as $n \to -\infty$. By the Intermediate Value Theorem, the function e^x takes on all positive values.

The absolute value of e^z is e^x, where x is the real part of z.

Now suppose that $e^{z_1} = e^{z_2}$, where z_1 and z_2 are in S. Then, using real and imaginary parts, we have

$$e^{x_1 - x_2} = e^{i(y_2 - y_1)}.$$

Since $e^{x_1 - x_2}$ is real and positive, and since $|y_1 - y_2| < 2\pi$ we must have $y_1 - y_2 = 0$, and hence $x_1 - x_2 = 0$.

The equation $e^z e^{-z} = 1$ shows that zero cannot be a value.

Suppose that $w \ne 0$. Then, as we have seen, $|w| = e^x$ for some real x. Then $w/|w|$ is on the unit circle, so there is $y \in [0, 2\pi)$ such that $w/|w| = e^{iy}$. Then $e^{x+iy} = |w|(w/|w|) = w$.

The lines in S that are parallel to the real axis are mapped onto rays emanating from, but not including, the origin.

An Example, as an application

The example gives functions f and g, both differentiable on $(0, \infty)$, with $g(x) \to +\infty$ as $x \to +\infty$, $g'(x) \ne 0$ in $(0, \infty)$, $f(x)/g(x) \to 1/2$ as $x \to +\infty$ but with $f'(x)/g'(x)$ having no limit as $x \to +\infty$. Theorem 1 will be applied to show the lack of limit.

We set $f(x) := x^2$, $g(x) := \sin x^2 + 2x^2$.
Then $f(x)/g(x) = x^2/x^2([\sin x^4/x^2] + 2) \to 1/2$ at $+\infty$.
Next, $g(x) = x^2([\sin x^4/x^2] + 2) \to +\infty$ and $g'(x) = 2x(\cos x^2 + 2) \ne 0$ on $(0, \infty)$.
By Theorem 1, $\cos x^2$ oscillates between 1 and -1 infinitely often as $x \to +\infty$ and

$$f'(x)/g'(x) = \frac{1}{\cos x^2 + 2}, \quad \text{so} \quad \limsup_{x \to +\infty} f'(x)/g'(x) = 1 \quad \text{and} \quad \liminf_{x \to +\infty} f'(x)/g'(x) = 1/3.$$

Thus $f'(x)/g'(x)$ has no limit as $x \to +\infty$. We observe that, in this example,

$$\liminf_{x \to +\infty} \frac{f'(x)}{g'(x)} < \lim_{x \to +\infty} \frac{f(x)}{g(x)} < \limsup_{x \to +\infty} \frac{f'(x)}{g'(x)}.$$

More applications

We can now use the exponential and trigonometric functions in **5.6**, **5.17**, **5.18** and the Exercises as remembered from Calculus, having derived their properties. It remains to check that the ratio of a circle's circumference to its diameter is π. We have to define what *circumference* means as part of that!