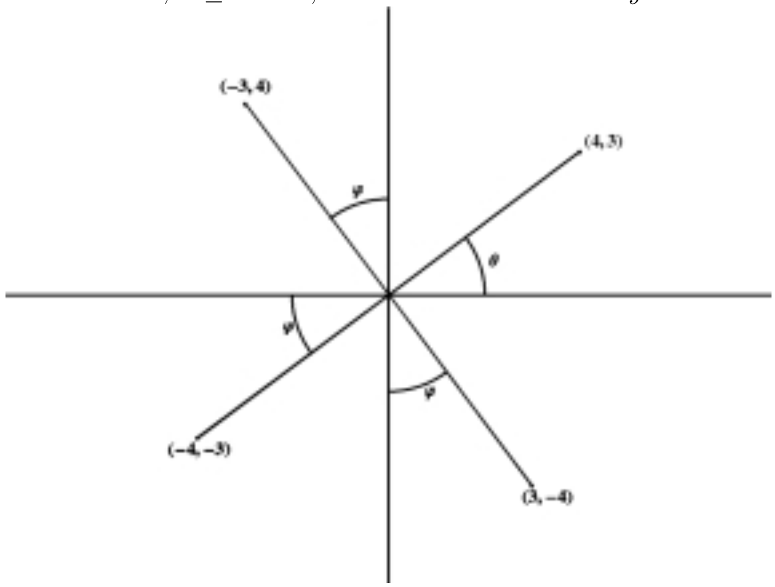
In $\mathbb{R}^2 \setminus \{(0,0)\}$ we can assign coordinates to a point $\mathbf{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ that tell us how far away from $(0,0)$ $\mathbf{p}$ is and what angle $\mathbf{p}$ makes with the positive x-axis, as long as the angle we mean is at least zero radians and is less than 2π radians. We seek $r > 0$ and θ, $0 \le \theta < 2\pi$, so that $x = r\cos\theta$ and $y = r\sin\theta$.



Calculating r is easy: $r = \sqrt{x^2 + y^2}$. Finding θ is complicated. We will divide $\mathbb{R}^2 \setminus \{(0,0)\}$ into four *quadrants,* the positive x axis, the negative x axis and likewise the positive and negative y axes.

In the first quadrant we have $x > 0$ and $y > 0$. The tangent of the angle between $\mathbf{p}$ and the positive x-axis is y/x so $\theta = \arctan(y/x)$.

In the second quadrant we have $x < 0$ and $y > 0$. Now we consider the right triangle with vertices $\mathbf{0}$, $(0, y)$ and $\mathbf{p}$ and the angle φ between $\mathbf{p}$ and the positive y-axis. We have $\tan\varphi = (-x)/y$ now. When we add φ to $\pi/2$ we arrive at the angle θ between $\mathbf{p}$ and the positive x-axis. Thus in quadrant two, $\theta = \arctan(-x/y) + \pi/2$.

In the third quadrant we have $x < 0$ and $y < 0$. We need to add π, now to the angle φ between $\mathbf{p}$ and the negative x-axis, to find θ. The tangent of φ is $(-y)/(-x) = y/x$, so in quadrant three $\theta = \arctan(y/x) + \pi$.

In the fourth quadrant we have $x > 0$ and $y < 0$. This time we add $3\pi/2$ to the angle φ between $\mathbf{p}$ and the negative y-axis. We have $\tan\varphi = x/(-y)$ so that
in the fourth quadrant, $\theta = \arctan(-x/y) + 3\pi/2$.

If $y = 0$, so that $\mathbf{p}$ lies along the x axis, $\theta = 0$ if $x > 0$ and $\theta = \pi$ if $x < 0$. Noticing that x is in the denominator in the formulas for quadrants one and three, we see that putting $y = 0$ there gives us the correct results.

If $x = 0$, so that $\mathbf{p}$ lies along the y axis, $\theta = \pi/2$ if $y > 0$ and $\theta = 3\pi/2$ if $y < 0$. Noticing that y is in the denominator in the formulas for quadrants two and four, we see that putting $x = 0$ there gives us the correct results.

Now let's go back and redefine the quadrants, so that they contain their "starting" rays but not their "ending" ones. We can display our results in tabular form:

Polar Coordinate Angles		
Quadrant	(meaning of)	θ
I	$x > 0,\ y \ge 0$	$\arctan(y/x)$
II	$x \le 0,\ y > 0$	$\arctan(-x/y) + \pi/2$
III	$x < 0,\ y \le 0$	$\arctan(y/x) + \pi$
IV	$x \ge 0,\ y < 0$	$\arctan(-x/y) + 3\pi/2$