Math 8601, Fall 2001 A modified proof of (1.9)

Perfect sets are uncountable

Introduction (can be skipped): Given a convergent sequence (for us, of distinct points), we can remove a *finite* number of terms and the new sequence still converges to the original limit. When we pass to the notion of *limit point* the idea is similar. The definition of limit point allows us to construct by induction a sequence of distinct points (belonging to the set under discussion) that converges to the limit point (Let E denote the set and y the limit point. We find $x_1 \in E$ with $0 < d_1 := |x_1 - y| < 1 =: d_0$. We then find $x_2 \in E$ with $0 < d_2 := |x_2 - y| < d_1/2$. Having found $x_k,\ 1 \le k \le K$ such that $0 < d_k := |x_k - y| < d_{k-1}/2,\ 1 \le k \le K$ we can find $x_{k+1} \in E$ with $0 < d_{k+1} := |x_{k+1} - y| < d_k/2$. This gives a sequence of distinct points, all different than y, that (because we divided by two at each step) converges to y). We can still remove any finite number of points from this sequence and the new sequence converges to the limit point.

A modified proof of (1.9) We suppose that $P = \{c_1, c_2, \ldots, c_n, \ldots\}$ is a perfect set. We seek a contradition. We observe that for every $y \in P$, we can remove any *finite* subset of P and it is still true that y is a limit point of the new set. We begin by choosing a point $y_1 \in P$ such that $y_1 \ne c_1$. Then $c_1 \notin B_{|y_1 - c_1|}(y_1)$ so for any $0 < d_1 < |y_1 - c_1|$ the closed ball $\bar{B}_1 := \overline{B_{d_1}(y_1)} = \{x \in \mathbb{R}^n : |x - y_1| \le d_1\}$ (the last equality requires proof!) does not contain c_1.

We are ready now to involve c_2. Since $y_1 \in P$, y_1 is a limit point of $P \setminus \{c_1, c_2\}$, so there exists $y_2 \in P \setminus \{c_1, c_2\}$ such that y_2 belongs to the *open* ball $B_1 := \{x \in \mathbb{R}^n : |x - y_1| < d_1\}$ and $y_2 \ne y_1$. By *construction,* $y_2 \ne c_2$. Then $c_2 \notin B_{|y_2 - c_2|}(y_2)$ so for any $0 < d_2 < |y_2 - c_2|$ the closed ball $\overline{B_{d_2}(y_2)}$ does not contain c_2. To define $\bar{B}_2$ we impose another restriction on d_2: we require $\overline{B_{d_2}(y_2)} \subseteq \bar{B}_1$. (This can be achieved by requiring that $0 < d_2 < \min\{|y_2 - c_2|, d_1 - |y_2 - y_1|\}$. We have $d_1 - |y_2 - y_1| > 0$ because y_2 is in the open ball B_1, whose closure is $\bar{B}_1$.) We now define $\bar{B}_2 := \overline{B_{d_2}(y_2)}$. We intend to use the open ball $B_2 := B_{d_2}(y_2)$ as well.

We now have $c_k \notin \bar{B}_k$ and $\bar{B}_k \cap P$ is compact if $1 \le k \le 2$. Moreover, $\bar{B}_\ell \subseteq \bar{B}_k$ if $1 \le \ell < k \le 2$.

(To understand the induction to follow, you might want to reread the paragraph involving c_2, increasing all the subscripts by one and replacing $P \setminus \{c_1, c_2\}$ by $P \setminus \{c_1, c_2, c_3\}$.)

$(*_K)$ Suppose we have selected closed balls $\bar{B}_k,\ 1 \le k \le K$, with centers $y_k \in P$, such that $c_k \notin \bar{B}_k$ and $\bar{B}_k \cap P$ is compact if $1 \le k \le K$, and such that $\bar{B}_\ell \subseteq \bar{B}_k$ if $1 \le \ell < k \le K$.

(What follows is a copy of the c_2 paragraph, modified)

Since $y_K \in P$, y_K is a limit point of $P \setminus \{c_1, c_2, \ldots, c_K\}$, so there exists $y_{K+1} \in P \setminus \{c_1, c_2, \ldots, c_K\}$ such that y_{K+1} belongs to the *open* ball $B_K := \{x \in \mathbb{R}^n : |x - y_K| < d_K\}$ and $y_{K+1} \ne y_K$. By *construction,* $y_{K+1} \ne c_{K+1}$. Then $c_{K+1} \notin B_{|y_{K+1} - c_{K+1}|}(y_{K+1})$ so for any $0 < d_{K+1} < |y_{K+1} - c_{K+1}|$ the closed ball $\overline{B_{d_{K+1}}(y_{K+1})}$ does not contain c_{K+1}. To define $\bar{B}_{K+1}$ we impose another restriction on d_{K+1}: we require $\overline{B_{d_{K+1}}(y_{K+1})} \subseteq \bar{B}_K$. This can be achieved by requiring that $0 < d_{K+1} < \min\{|y_{K+1} - c_{K+1}|, d_K - |y_{K+1} - y_K|\}$. We have $d_K - |y_{K+1} - y_K| > 0$ because y_{K+1} is in the open ball B_K, whose closure is $\bar{B}_K$. We have obtained the truth of $(*_{K+1})$.

Conclusion $\{\bar{B}_k \cap P\}$ is a decreasing sequence of compact nonempty sets so the intersection of these sets is a nonempty subset of P. This is a contradiction, for $c_k \notin \bar{B}_k \cap P$. Hence if P is perfect, P is uncountable.