Problem 2: Due Feb 13

Suppose that $m \in \mathbb{N}$ and $n \in \mathbb{N}$ and $k \in \mathbb{N}$ and $k \neq 0$. Prove that, if $m = n + k$ then $n \neq m + k$.

Remark: This is part of the proof of the Trichotomy Law. Thus we cannot use the Trichotomy Law in the proof.

Solution: The quantifiers here are "for all" for each of the three variables. Thus we assume m and n and k are given, and arbitrary, except that $k \neq 0$ (we could write our statement as:

$$(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(\forall k \in \mathbb{Z}^+)(m = n + k \Rightarrow n \neq m + k), \quad \text{but we don't have to here).}$$

If $m \neq n + k$ there is nothing to prove, by vacuity.

If $m = n + k$ we have to prove that $n \neq m + k$.

We use contradiction, assuming that $n = m + k$.

By substitution $n = m + k = (n + k) + k = n + (k + k)$ (the last by associativity).

Then $n = n + 0 = n + (k + k)$,

so by cancellation $0 = k + k$.

Since $k \neq 0$, there exists $j \in \mathbb{N}$ such that $k = s(j) = j + 1$.

Then $0 = k + k = k + s(j) = k + (j + 1) = (k + j) + 1 = s(k + j)$.

This contradicts the Axiom that $0 = s(j)$ is false for all $j \in \mathbb{N}$.

This completes the proof that $n \neq m + k$.

Since m and n in $\mathbb{N}$ were arbitrary and $k \in \mathbb{Z}^+$ is arbitrary, "$m = n + k \Rightarrow n \neq m + k$" is true for all the indicated variables, and the proof is complete.