An addendum to the Exponential Saga

One of the things we want to prove in this note is the power series formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \text{ We will for now define } E(z) \text{ by } E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!},\ z \in \mathbb{C}.$$

We will show first that for $x \geq 0$, $e^x = E(x)$. Then we will prove some Theorems about *double* series. These Theorems will allow us to prove that $E(x) = e^x$ for all real x. Theorems about double series will also be used in the proof of an important and possibly startling fact: if two power series agree on a sequence that converges to a point *within* the circle of convergence, then they have the same coefficients, and so agree everywhere inside their circle of convergence.

Proof of equality for non-negative x.

If $x = 0$, $E(x) = E(2) = 1 = e^0 = e^x$, so equality holds in this case. If $x > 0$, we will go back to the definition of e^x as

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

and recall (2.2) there:

$$\left(1 + \frac{x}{n}\right)^n = \sum_{r=0}^{n} \frac{n!}{(n-r)!n^r} \frac{x^r}{r!} < \sum_{r=0}^{n} \frac{x^r}{r!}.$$

Since $x > 0$, $\left(1 + \frac{x}{n}\right)^n < \sum_{r=0}^{n} \frac{x^r}{r!} < E(x)$. We drop the middle expression, let $n \to \infty$, and obtain the result that $e^x = f(x) \leq E(x)$.

The inequality in the other direction is more complicated, and the argument is harder to think of. We have

$$\left(1 + \frac{x}{n}\right)^n = \sum_{r=0}^{n} \frac{n!}{(n-r)!n^r} \frac{x^r}{r!}.$$

The quantities $\frac{n!}{(n-r)!n^r}$ stand between us and the 1 we want in place of them. First we choose some $N \in \mathbb{N}_1$ and we ask that $n > 2N+1$. A closer look at the quantities $\frac{n!}{(n-r)!n^r}$, when $r \geq 2$ gives

$$\frac{n!}{(n-r)!n^r} = \frac{(n-1)!}{(n-r)!n^{r-1}} = \frac{1}{n^{r-1}} \prod_{k=1}^{r-1} (n-k) = \prod_{k=1}^{r-1} \frac{(n-k)}{n} = \prod_{k=1}^{r-1} \left(1 - \frac{k}{n}\right).$$

Therefore we have

$$\left(1 + \frac{x}{n}\right)^n = 1 + x + \sum_{r=2}^{n} \frac{n!}{(n-r)!n^r} \frac{x^r}{r!} > 1 + x + \sum_{r=2}^{N+1} \frac{n!}{(n-r)!n^r} \frac{x^r}{r!} = 1 + x + \sum_{r=2}^{N+1} \frac{x^r}{r!} \prod_{k=1}^{r-1} \left(1 - \frac{k}{n}\right).$$

Now since $k < r \leq N+1$, if we replace each $\left(1 - \frac{k}{n}\right)$ by $\left(1 - \frac{N}{n}\right)$ the big product

$$\prod_{k=1}^{r-1} \left(1 - \frac{k}{n}\right) > \prod_{k=1}^{r-1} \left(1 - \frac{N}{n}\right) = \left(1 - \frac{N}{n}\right)^{r-1} > \left(1 - \frac{N}{n}\right)^N,$$

because each factor is less than 1. We can put this factor onto the $1+x$ as well, and the right-hand side quantity becomes smaller yet. We are close to the desired result now! We have shown that if $n > 2N+1$ then

$$\left(1 + \frac{x}{n}\right)^n > \left(1 - \frac{N}{n}\right)^N \sum_{r=0}^{N+1} \frac{x^r}{r!}.$$

Next we let $n \to \infty$. The left-hand side converrges to e^x and the factor $\left(1 - \frac{N}{n}\right)^N$ converges to 1, so

$$e^x \geq \sum_{r=0}^{N+1} \frac{x^r}{r!}.$$

Finally, we let $N \to \infty$ and we get $e^x \geq E(x)$. We have proved the inequality in the other direction, so the proof that $e^x = E(x)$ for $x \geq 0$ is complete.

To prove that $e^x = E(x)$ for negative x as well, we will show that, like e^x, $E(x)$ has the property $E(x+y) = E(x)E(y)$ for all real x and y.

But then, for $x > 0$, $e^x e^{-x} = 1 = E(2) = E(x)E(-x) = e^x E(-x)$, and now we cancel the factors e^x and have $e^{-x} = E(-x)$. This will complete the proof that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all real } x.$$

Thus now we have to prove the product formula. Actually, we'll prove that $E(z+w) = E(z)E(w)$ for all complex z and w. This means we have to prove that

$$E(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{n=0}^{\infty} \frac{w^n}{n!} = E(z)E(w) \quad \text{for all complex } z \text{ and } w. \tag{1}$$

The series on the left is

$$\sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \frac{z^{n-r} w^r}{n!}$$

We want to interchange the order of summation, because it turns out that if we do so, and make a suitable change of indices, the result is exactly what we want, the product on the right-hand side of (1).

We will now make a long digression! Theorems at locations (2) and (8) will be used at (9) to prove (1).

Introduction to double series

In this note, because we are working with power series in mind, "natural numbers" will refer to $\mathbb{N}_0$.

A *double series* has terms with two subscripts instead of one. If the terms are a_{nm}, we think of the terms a_{nm} arranged in a rectangular array, with a_{nm} assigned to the point (n, m). We regard the first subscript (here n) as the "row index," so that $a_{00}, a_{10}, a_{20}, \ldots, a_{n\,0}, \ldots$ occupy the "bottom row" of our first-quadrant array. The second subscript (here m) is the "column index," so that $a_{00}, a_{01}, a_{02}, \ldots, a_{0\,n}, \ldots$ occupy the "leftmost column" of our first-quadrant array.

For a double series we choose whether to "add by rows" or "add by columns." That is, we choose between

$$\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{nm} \right) \text{ (add by rows)} \quad \text{and} \quad \sum_{m=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{nm} \right) \text{ (add by columns).}$$

What we have here is a pair of series consisting of terms that are themselves series. The partial sums for the first (add by rows) series are

$$R_M := \sum_{m=0}^{M} \left(\sum_{n=0}^{\infty} a_{nm} \right) = \sum_{m=0}^{M} r_m, \quad \text{where } r_m := \sum_{n=0}^{\infty} a_{nm}.$$

The partial sums for the second (add by columns) series are

$$C_N := \sum_{n=0}^{N} \left(\sum_{m=0}^{\infty} a_{nm} \right) = \sum_{n=0}^{N} c_n, \quad \text{where } c_n := \sum_{m=0}^{\infty} a_{nm}.$$

The sum r_m is the "sum" along row m. Each row is named by the column index, here m, which remains constant in that row. Similarly, c_n is the "sum" along column n.

We define convergence, for each way of summing, in the usual way – by requiring the appropriate (row or column) sequence of partial sums to converge. Otherwise, we say the double series diverges by rows or by columns.

A very important question arises at once: does the order of summation make any difference? The answer is "Yes!" and here is an example to prove it:

$$\text{Let } a_{kk} = 1,\ a_{k,k+1} = -1 \text{ for all } k \in \mathbb{N}_0, \text{ and let } a_{nm} = 0 \text{ otherwise.}$$

Then $r_m = 0$ for all m and $c_0 = 1$, while $c_n = 0$ when $n > 0$. Thus

$$0 = \sum_{m=0}^{\infty} r_m = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{nm}\right) \quad \text{and} \quad \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{nm}\right) = \sum_{n=0}^{\infty} c_n = 1.$$

Thus both double sums are finite but

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{nm}\right) \neq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{nm}\right).$$

Another worrisome feature of this example is that the terms in the series do not tend to zero!

However, not all is lost. Difficulties do not occur if all the terms are non-negative!

The double-series theorem for double series with non-negative terms

(2) **Theorem:** *If* $p_{nm} \geq 0$ *for every pair* (m, n) *of natural numbers, then*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} p_{nm}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} p_{nm}\right).$$

Remark: The equation in the statement of the Theorem is true whether or not the double sums are finite!

Proof: If both series diverge to $+\infty$, there is nothing to show: the equation holds!

Suppose that one of the sums is finite. The first case is that the series added by columns is finite. We are assuming:

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} p_{nm}\right) < +\infty.$$

Somehow we have to "reduce" the problem to changing the order of a *finite* double sum. The really important (hard-to-see) idea is to use the simple inequalities

$$p_{nM} \leq \sum_{m=0}^{\infty} p_{nm} = c_n, \quad \text{for each } M \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}_0, \quad \text{since } p_{nM} \text{ is a term in the sum } \sum_{m=0}^{\infty} p_{nm}.$$

These inequalities, for each fixed $M \in \mathbb{N}_0$, and the Comparison Test yield that (now putting $M = m$),

$$p_{nm} \leq c_n \text{ and } \sum c_n \text{ converges, so } r_m = \sum_{n=0}^{\infty} p_{nm} \leq \sum_{n=0}^{\infty} c_n < +\infty \text{ for each } m \in \mathbb{N}_0.$$

In other words, each *row* sum is finite (convergent) because the series summed by columns is convergent. Now we have to prove that $\sum_{m=0}^{\infty} r_m < +\infty$ (converges). Since the terms r_m are non-negative, all we have to do is to show that the partial sums $R_M = \sum_{m=0}^{M} r_m$ are bounded above.

It will be to our advantage if we show that the partial sums are bounded above by the sum by columns, namely

$$C := \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} p_{nm}\right) < +\infty. \tag{3}$$

We will actually only show that for every $\epsilon > 0$, $R_M = \sum_{m=0}^{M} r_m < C+\epsilon$. But then we will know that $\sum_{m=0}^{\infty} r_m \le C+\epsilon$ for all $\epsilon > 0$, and hence (by an argument that you should be able to do by now) $\sum_{m=0}^{\infty} r_m \le C$.

First we allow M to be a given natural number, no matter how large, and we allow $\epsilon > 0$ to be given, no matter how small. Since $r_m = \sum_{n=0}^{\infty} p_{nm} < +\infty$ for each $m \in \mathbb{N}_0$, we can find, (for each $m \in \mathbb{N}_0$) a natural number N_m such that

$$r_m = \sum_{n=0}^{\infty} p_{nm} < \left(\sum_{n=0}^{N_m} p_{nm}\right) + \epsilon/(M+1).$$

This only needs to be done for $0 \le m \le M$. Now we define $N_\epsilon := \max\{N_0, N_1, \cdots, N_M\}$. Then because $p_{nm} \ge 0$, we can replace each N_m by the (probably) larger quantity N_ϵ:

$$r_m = \sum_{n=0}^{\infty} p_{nm} < \left(\sum_{n=0}^{N_\epsilon} p_{nm}\right) + \epsilon/(M+1),\ 0 \le m \le M.$$

Therefore (we'll drop the parentheses about the sum in n now)

$$\begin{aligned} R_M = \sum_{m=0}^{M} r_m &< \sum_{m=0}^{M}\left(\sum_{n=0}^{N_\epsilon} p_{nm} + \epsilon/(M+1)\right) \\ &= \sum_{m=0}^{M}\sum_{n=0}^{N_\epsilon} p_{nm} + \sum_{m=0}^{M} \epsilon/(M+1) \\ &= \sum_{m=0}^{M}\sum_{n=0}^{N_\epsilon} p_{nm} + \epsilon \\ &= \sum_{n=0}^{N_\epsilon}\sum_{m=0}^{M} p_{nm} + \epsilon. \end{aligned}$$

At last, we have used the fact that we can rearrange finite sums! Therefore in the inner sum we can replace M by $+\infty$, because that can only increase the terms. This gives, from what we just did,

$$\begin{aligned} R_M &< \sum_{n=0}^{N_\epsilon}\sum_{m=0}^{M} p_{nm} + \epsilon \\ &\le \sum_{n=0}^{N_\epsilon}\sum_{m=0}^{\infty} p_{nm} + \epsilon \\ &= \sum_{n=0}^{N_\epsilon} c_n + \epsilon = C_{N_\epsilon} + \epsilon \le C + \epsilon. \end{aligned}$$

Thus [recall the discussion just after (3)]

$$R_M < C+\epsilon, \ \text{ so } \ R := \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} p_{nm}\right) = \sum_{m=0}^{\infty} r_m = \lim_{M\to\infty} R_M \le C+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, $R \le C$. We have shown that if $C < \infty$, then $R < \infty$, and we have even shown that $R \le C$. But now that we know $R < \infty$, we can redo the argument, with the roles of row and column reversed (I highly recommend that you go through the steps!) and obtain the estimate $C \le R$.

Had it been R that was finite, we would have had to carry out the reversed-rôle argument first. The result would have been the same. Thus the case that one sum be infinite and the other be finite cannot occur. This completes the proof of the Theorem.

We can now define absolute convergence, in the natural way, for double series, and we note that we can use the more convenient order of summing, if there is one.

Absolute convergence for double series with complex terms

Definition: *A double series* $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_{nm}$ *converges absolutely if* $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |z_{nm}|$ *converges.*

The next natural questions: Does absolute convergence imply convergence? And in that case, does the order of summation matter? The answers are: "Yes!" and "No!" respectively.

The Absolute Convergence Theorem for double series with complex terms

(4) **Theorem:** *If the double series*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_{nm}$$

with complex terms converges absolutely, then both series

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} z_{nm}\right) \quad \text{and} \quad \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right) \tag{5}$$

converge absolutely and converge. They have the same sum (value, limit).

Proof: Sometimes it will be handy to use a short name for a complicated expression. Let's again let

$$r_m := \sum_{n=0}^{\infty} z_{nm} \quad \text{(sum along row } m\text{)}, \quad \text{and} \quad c_n := \sum_{m=0}^{\infty} z_{nm} \quad \text{(sum along column } n\text{)}.$$

We'll prove the absolute convergence and convergence statements first. Our proof will actually have to include the proof that a complex *single* series converges if it converges absolutely. We are given that $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |z_{nm}|$ converges. We express this (because the terms are non-negative) as: $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |z_{nm}| < \infty$. Since $|c_n| = \left|\sum_{m=0}^{\infty} z_{nm}\right| \le \sum_{m=0}^{\infty} |z_{nm}| < \infty$ for each n,

$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{\infty}\left|\sum_{m=0}^{\infty} z_{nm}\right| \le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |z_{nm}| < \infty.$$

This, and a similar argument for row sums, already shows that both series in (5) converge absolutely.

(6) [Please recall that $|\sum_{m=0}^{\infty} z_{nm}| \le \sum_{m=0}^{\infty} |z_{nm}|$ follows from $\left|\sum_{m=0}^{M} z_{nm}\right| \le \sum_{m=0}^{M} |z_{nm}|$ (which is the triangle inequality for complex numbers, plus induction) and from taking the limit as $M \to \infty$, together with the continuity of the absolute-value function $z \mapsto |z|$.]

For any complex number z we can write $z = x + iy$, where the real number x is the real part of z and the real number y is the imaginary part of z. Then

$$\max\{x^2, y^2\} \le x^2 + y^2 = |z|^2 \le x^2 + 2|x||y| + y^2 = (|x| + |y|)^2,$$

so (please double check!)

$$\frac{|x|+|y|}{2} \le \max\{|x|,\,|y|\} \le |z| \le |x|+|y|. \tag{7}$$

If we now write our $c_n =: u_n + iv_n$, the inequalities (7) that we just did show that $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=0}^{\infty} v_n$ both converge absolutely, hence they converge. But then the sequence of partial sums of $\sum_{n=0}^{\infty} c_n$ has terms $U_n + iV_n$, where $U_n := \sum_{k=0}^{n} u_k$ and $V_n := \sum_{k=0}^{n} v_k$, so the sequence of partial sums of $\sum_{n=0}^{\infty} c_n$ converges, and thus the series converges. The same argument, with n replaced by m and c by r shows that both series

$$\sum_{m=0}^{\infty} r_m = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} z_{nm}\right) \quad\text{and}\quad \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right)$$

converge.

Next we must prove that the two series converge to the same limit. The double series converges absolutely, that is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |z_{nm}| < \infty.$$

Therefore by (2), for any $\epsilon > 0$ there exist $N \in \mathbb{N}_0$ and $M \in \mathbb{N}_0$ such that

$$\sum_{n=N+1}^{\infty}\left(\sum_{m=0}^{\infty} |z_{nm}|\right) < \epsilon/4 \quad\text{and}\quad \sum_{m=M+1}^{\infty}\left(\sum_{n=0}^{\infty} |z_{nm}|\right) < \epsilon/4.$$

Now let's rewrite RC to take advantage of these $\epsilon/4$'s:

$$RC := \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right) = \sum_{n=0}^{N}\left(\sum_{m=0}^{\infty} z_{nm}\right) + \sum_{n=N+1}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right).$$

We'll rewrite the first sum on the right-hand side as

$$\sum_{n=0}^{N}\left(\sum_{m=0}^{\infty} z_{nm}\right) = \sum_{n=0}^{N}\left(\sum_{m=0}^{M} z_{nm} + \sum_{m=M+1}^{\infty} z_{nm}\right) = \sum_{n=0}^{N}\left(\sum_{m=0}^{M} z_{nm}\right) + \sum_{n=0}^{N}\left(\sum_{m=M+1}^{\infty} z_{nm}\right).$$

We now substitute this new form into the expression for RC:

$$RC = \sum_{n=0}^{N}\left(\sum_{m=0}^{M} z_{nm}\right) + \sum_{n=0}^{N}\left(\sum_{m=M+1}^{\infty} z_{nm}\right) + \sum_{n=N+1}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right).$$

In a similar way (you should go through the steps yourself!) we can write

$$CR := \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} z_{nm}\right)$$

in the form

$$CR = \sum_{m=0}^{M}\left(\sum_{n=0}^{N} z_{nm}\right) + \sum_{m=0}^{M}\left(\sum_{n=N+1}^{\infty} z_{nm}\right) + \sum_{m=M+1}^{\infty}\left(\sum_{n=0}^{\infty} z_{nm}\right).$$

Because they are finite sums,

$$\sum_{m=0}^{M}\left(\sum_{n=0}^{N} z_{nm}\right)=\sum_{n=0}^{N}\left(\sum_{m=0}^{M} z_{nm}\right).$$

This gives

$$\begin{aligned} RC-CR=&\sum_{n=0}^{N}\left(\sum_{m=M+1}^{\infty} z_{nm}\right)+\sum_{n=N+1}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right)\\ &-\sum_{m=0}^{M}\left(\sum_{n=N+1}^{\infty} z_{nm}\right)-\sum_{m=M+1}^{\infty}\left(\sum_{n=0}^{\infty} z_{nm}\right). \end{aligned}$$

[[The next steps can be skipped, maybe; all they do is apply the triangle inequality and the "infinite triangle inequality" discussed in the [Please recall...] statement at location (6).]]

We now have

$$\begin{aligned} |RC-CR|\le&\left|\sum_{n=0}^{N}\left(\sum_{m=M+1}^{\infty} z_{nm}\right)\right|+\left|\sum_{n=N+1}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right)\right|\\ &+\left|\sum_{m=0}^{M}\left(\sum_{n=N+1}^{\infty} z_{nm}\right)\right|+\left|\sum_{m=M+1}^{\infty}\left(\sum_{n=0}^{\infty} z_{nm}\right)\right|\\ \le&\sum_{n=0}^{N}\left|\sum_{m=M+1}^{\infty} z_{nm}\right|+\sum_{n=N+1}^{\infty}\left|\sum_{m=0}^{\infty} z_{nm}\right|\\ &+\sum_{m=0}^{M}\left|\sum_{n=N+1}^{\infty} z_{nm}\right|+\sum_{m=M+1}^{\infty}\left|\sum_{n=0}^{\infty} z_{nm}\right|. \end{aligned}$$

Then, by the "infinite triangle inequality" discussed at (6),

$$\begin{aligned} |RC-CR|\le&\sum_{n=0}^{N}\sum_{m=M+1}^{\infty}|z_{nm}|+\sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty}|z_{nm}|\\ &+\sum_{m=0}^{M}\sum_{n=N+1}^{\infty}|z_{nm}|+\sum_{m=M+1}^{\infty}\sum_{n=0}^{\infty}|z_{nm}|\\ =&\sum_{m=M+1}^{\infty}\sum_{n=0}^{N}|z_{nm}|+\sum_{n=N+1}^{\infty}\sum_{m=0}^{\infty}|z_{nm}|\\ &+\sum_{n=N+1}^{\infty}\sum_{m=0}^{M}|z_{nm}|+\sum_{m=M+1}^{\infty}\sum_{n=0}^{\infty}|z_{nm}|. \end{aligned}$$

There are four sums after the equality sign. The second and fourth ones (reading left-to-right and top-to-bottom) are bounded by $\epsilon/4$. The first and third are bounded by the fourth and second terms, respectively. All together then $|RC-CR|<\epsilon$. Since $\epsilon>0$ is arbitrary, $CR=RC$. That is, we have completed the proof that

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} z_{nm}\right)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} z_{nm}\right)$$

if the double series converges absolutely. Let us remember that we can sum the series of absolute values in either order. One order might well be more convenient than the other!

(8) **Proof of the product rule for the function** $E(z)$

We can now prove the statement at (1):

$$E(z+w)=\sum_{n=0}^{\infty}\frac{(z+w)^n}{n!}=\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{n=0}^{\infty}\frac{w^n}{n!}=E(z)E(w)\ \text{ for all complex } z \text{ and } w.$$

We have

$$\sum_{n=0}^{\infty}\frac{(z+w)^n}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{z^{n-r}w^r}{n!},$$

which is not exactly a double series. But we can introduce a notation due to Iverson and Knuth (Donald Knuth, *Two notes on Notation, American Mathematical Monthly,* vol. 99, 1992, pp 403–426) that will give us a double series with lots of zero terms.

Let us define a function of one or more variables as follows:

$$[\ \text{A:= statement about the variables}\ ] := \begin{cases} 1 & \text{if A is true} \\ 0 & \text{if A is false.}\end{cases}$$

We will use

$$[r\le n] := \begin{cases} 1 & \text{if } r\le n \\ 0 & \text{if } r>n.\end{cases}$$

This lets us write

$$\sum_{n=0}^{\infty}\frac{(z+w)^n}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{z^{n-r}w^r}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}[r\le n]\binom{n}{r}\frac{z^{n-r}w^r}{n!}.$$

Does this double series converge absolutely? We put in absolute values and take a look!

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}[r\le n]\binom{n}{r}\frac{|z^{n-r}w^r|}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}[r\le n]\binom{n}{r}\frac{|z|^{n-r}|w|^r}{n!}=\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}[r\le n]\binom{n}{r}\frac{|z|^{n-r}|w|^r}{n!}.$$

We put the absolute values on z and w and changed the order of summation. The inner sum is

$$\sum_{n=0}^{\infty}[r\le n]\binom{n}{r}\frac{|z|^{n-r}|w|^r}{n!}.$$

The terms are $[r\le n]\binom{n}{r}\frac{|z|^{n-r}|w|^r}{n!}$. We can simplify by using the formula for $\binom{n}{r}$ and we get

$$[r\le n]\binom{n}{r}\frac{|z|^{n-r}|w|^r}{n!}=[r\le n]\frac{|z|^{n-r}}{(n-r)!}\frac{|w|^r}{r!}=\frac{|w|^r}{r!}\cdot[r\le n]\frac{|z|^{n-r}}{(n-r)!}.$$

The factor $\frac{|w|^r}{r!}$ does not depend on n, so it can be factored out of the series that is “the inner sum.” This, and realizing what $[r\le n]$ means gives

$$\sum_{n=0}^{\infty}[r\le n]\binom{n}{r}\frac{|z|^{n-r}|w|^r}{n!}=\frac{|w|^r}{r!}\sum_{n=0}^{\infty}[r\le n]\frac{|z|^{n-r}}{(n-r)!}=\frac{|w|^r}{r!}\sum_{n=r}^{\infty}\frac{|z|^{n-r}}{(n-r)!}.$$

We now can make the literal change of indices that replaces n by $n+r$:

$$\sum_{n=r}^{n=\infty}\frac{|z|^{n-r}}{(n-r)!}=\sum_{n+r=r}^{n+r=\infty}\frac{|z|^{n+r-r}}{(n+r-r)!}=\sum_{n=0}^{n=\infty}\frac{|z|^n}{n!}=\sum_{n=0}^{\infty}\frac{|z|^n}{n!}=E(|z|)=e^{|z|}<\infty.$$

We have therefore shown that

$$\sum_{n=0}^{\infty}[r\le n]\binom{n}{r}\frac{|z|^{n-r}|w|^r}{n!}=\frac{|w|^r}{r!}e^{|z|}.$$

This, in turn, means that

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}[r \le n]\binom{n}{r}\frac{|z^{n-r}w^r|}{n!} = \sum_{r=0}^{\infty}\frac{|w|^r}{r!}e^{|z|} = e^{|w|}e^{|z|} < \infty.$$

Therefore the series does converge absolutely. It can therefore be summed in either order. We choose the same order we used to check absolute convergence. This only amounts to removing all the absolute value signs! Thus we get as the result, not $= e^{|w|}e^{|z|}$ but $E(z)E(w)$, which is what we wanted:

$$E(z+w) = E(z)E(w).$$

Here is a Theorem about Power Series that uses the Absolute Convergence Theorem for double series with complex terms. It's called a "rearrangement theorem," but the meaning here is different than the Theorem about rearranging conditionally convergent series to get any given sum. The theorem *might* be called a "change of base point theorem" instead.

(9) **Theorem (Power Series Rearrangement):** *If the power series*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

converges for $|z| < R$, *and if* $|z_o| < R$, *then the power series*

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(z_o)}{n!}(z-z_o)^n$$

converges for all z *such that* $|z-z_o| < R-|z_o|$, *and for all such* z, *the sum of the series coincides with* $f(z)$. *That is,*

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_o)}{n!}(z-z_o)^n \quad \text{for all} \quad z \quad \text{such that} \quad |z-z_o| < R-|z_o|.$$

Remark: If $R = +\infty$, then $R-|z_o| = +\infty$, so the statement could be modified to make that point more explicit.

Proof of the Power Series Rearrangement Theorem: Let us assume that $|z_o| < R$ and that $|z-z_o| < R-|z_o|$, which really amounts to assuming that $R_1 := |z-z_o|+|z_o| < R$. We can subtract and add z_o inside the powers of z in the power series for $f(z)$, and use the Binomial Formula:

$$(10) \qquad f(z) = \sum_{n=0}^{\infty} a_n(z-z_o+z_o)^n = \sum_{n=0}^{\infty} a_n \sum_{r=0}^{n}\binom{n}{r} z_o^{n-r}(z-z_o)^r = \sum_{n=0}^{\infty}\sum_{r=0}^{n} a_n\binom{n}{r} z_o^{n-r}(z-z_o)^r.$$

The series on the right-hand side is not a full double series, but we know how to introduce zero terms that will make it so. We need not do so yet because we will not change the order of summation when we check for absolute convergence, which we will do now:

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}|a_n|\binom{n}{r}|z_o|^{n-r}|z-z_o|^r = \sum_{n=0}^{\infty}|a_n|\sum_{r=0}^{n}\binom{n}{r}|z_o|^{n-r}|z-z_o|^r = \sum_{n=0}^{\infty}|a_n|(|z_o|+|z-z_o|)^n.$$

At the start of the proof we defined $R_1 = |z-z_o|+|z_o|$, and we assumed that $R_1 < R$. We know that power series converge absolutely at any point strictly inside the circle of convergence (the circle whose radius is the radius of convergence of the series). Since $R_1 < R$, the point R_1 is strictly inside the circle of convergence. Therefore

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}|a_n|\binom{n}{r}|z_o|^{n-r}|z-z_o|^r = \sum_{n=0}^{\infty}|a_n|(|z_o|+|z-z_o|)^n = \sum_{n=0}^{\infty}|a_n|R_1^n < \infty.$$

Thus the last series in (10) converges absolutely, as a double series, and so the order of summation can be changed. Once again we will use the Iverson-Knuth notation, and use the function $[r \leq n]$, skipping some steps:

$$f(z) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} a_n \binom{n}{r} z_o^{n-r} (z - z_o)^r = \sum_{r=0}^{\infty} \left(\sum_{n=0}^{\infty} [r \leq n] a_n \binom{n}{r} z_o^{n-r} \right) (z - z_o)^r. \tag{11}$$

The inner sum can be rewritten now:

$$\sum_{n=0}^{\infty} [r \leq n] a_n \binom{n}{r} z_o^{n-r} = \frac{1}{r!} \sum_{n=r}^{\infty} a_n n(n-1) \cdots (n-r+1) z_o^{n-r}.$$

Since power series can be differentiated term-by-term, and $n(n-1)\cdots(n-r+1)z_o^{n-r}$ is the r-th derivative of z^n, taken at $z = z_o$, the inner sum is simply $\dfrac{f^{(r)}(z_o)}{r!}$. This gives (after substitutions)

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(z_o)}{r!} (z - z_o)^r. \tag{12}$$

This series converges absolutely, by the Absolute Convergence Theorem for double series with complex terms.

When do two power series have to have the same coefficients?

The answer is striking! If two power series agree on a sequence that converges to a limit z_o with $|z_o| < R$, their coefficients must all be the same. In an earlier version we assumed equality on a whole disc with center 0.

(13) **Theorem:** *If the power series*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) := \sum_{n=0}^{\infty} b_n z^n$$

both converge for $|z| < R$, and if there exists a complex number z_o with $|z_o| < R$, and a complex sequence $\{z_k\}$ such that $z_o \neq z_k \to z_o$, where $|z_k| < R$ for all k, and

$$\text{if } f(z_k) = g(z_k) \text{ for all } k, \text{ then } a_n = b_n \text{ for all } n \in \mathbb{N}_0.$$

Proof: The idea for this proof is taken from Theorem 8.5, *Principles of Mathematical Analysis, 3d ed.,* by Walter Rudin. The proof also uses the Power Series Rearrangement Theorem. It will help to reduce the work to recast the Theorem a little. We define

$$h(z) := f(z) - g(z) = \sum_{n=0}^{\infty} (a_n - b_n) z^n =: \sum_{n=0}^{\infty} c_n z^n, \quad \text{where } c_n = a_n - b_n \text{ for all } n \in \mathbb{N}_0.$$

a power series that converges for $|z| < R$. We have $h(z_k) = 0$ for all k. If we can show that $h(z) = 0$ for all $|z| < R$, then we can show that $c_n = 0$ for all $n \in \mathbb{N}_0$.

We will first prove that $h(z) = 0$ for all z such that $|z - z_o| < R - |z_o|$. This will prove the Theorem in case $R = +\infty$ by Taylor's Theorem, just as we did in class. Otherwise, we'll have to expand the circle and pull its center toward 0 to complete the proof. Please remember how this part of the proof works! We will be using the same argument again, perhaps many times.

We know that the function $h(z)$ is a continuous function of z. Therefore

$$h(z_o) = \lim_{n \to \infty} h(z_k) = 0.$$

We know that $h(z)$ is differentiable at z_o. Therefore

$$h'(z_o) = \lim_{z \to z_o} \frac{h(z) - h(z_o)}{z - z_o} = \lim_{k \to \infty} \frac{h(z_k) - h(z_o)}{z_k - z_o} = 0.$$

By the Power Series Rearrangement Theorem, (9),

$$h(z)=\sum_{n=0}^{\infty}\frac{h^{(n)}(z_o)}{n!}(z-z_o)^n \text{ for all } z \text{ such that } |z-z_o|<R-|z_o|.$$

Since $h(z_o)=0=h'(z_o)$, we know that, at least,

$$h(z)=\sum_{n=2}^{\infty}\frac{h^{(n)}(z_o)}{n!}(z-z_o)^n \text{ for all } z \text{ such that } |z-z_o|<R-|z_o|.$$

We will show that all of the coefficients in this power series are zero. This won't complete the proof, but it will allow us to start a process that will prove the Theorem!

Suppose not, namely suppose that $H_n := \frac{h^{(n)}(z_o)}{n!} \neq 0$ for some n. Then there will be a least such n by the Well-Ordering Principle. We give this least n the name N. We have already seen that $N \geq 2$. Then for all z such that $|z-z_o|<R-|z_o|$,

$$h(z)=\sum_{n=N}^{\infty}H_n(z-z_o)^n=(z-z_o)^N\sum_{n=N}^{\infty}H_n(z-z_o)^{n-N}=(z-z_o)^N\sum_{n=0}^{\infty}H_{n+N}(z-z_o)^n,$$

the power series $g(z):=\sum_{n=0}^{\infty}H_{n+N}(z-z_o)^n$ is continuous at z_o, and $g(z_o)=H_N\neq 0$. Then there is a $\delta>0$ such that $|z-z_o|<\delta$ implies that $g(z)\neq 0$. Hence $h(z)=(z-z_o)^N g(z)\neq 0$ if $0<|z-z_o|<\delta$. But there exist infinitely many z_k such that $0<|z_k-z_o|<\delta$, since $z_k\to z_o$. For each k however $h(z_k)=0$. But when $|z_k-z_o|<\delta$ we must have $h(z_k)\neq 0$. This is a contradiction. Therefore we have proved that all the $H_n=0$. In other words,

$$\text{for all } z \text{ such that } |z-z_o|<R-|z_o|,\ h(z)=0.$$

In case $R<+\infty$, we have "taken a circular bite out of the circle" by showing that if $|z-z_o|<R-|z_o|$, then $h(z)=0$. We want to expand this "circle of zero-ness." As we noted at the outset, if R is infinite, we are done, for then $h(z)=0$ for all z and then, by Taylor's Theorem, all the coefficients $c_n=a_n-b_n=0$.

Thus we suppose that $R<+\infty$, and define $r:=R-|z_o|$. There are two possibilities: $r\geq|z_o|$ and $r<|z_o|$.

First possibility: $r\geq|z_o|$

If $r\geq|z_o|$ this means that 0 lies in the set $\{z\in\mathbb{C}:|z-z_o|\leq r\}$. We can then construct a sequence $Z_k\to 0$ such that $h(Z_k)=0$. We now let $Z_o=0$, and use the argument we used on z_o, getting:

$$h(z)=0 \text{ for all } z \text{ such that } |z-Z_o|<R-|Z_o|$$

Since $Z_o=0$, we have shown

$$h(z)=0 \text{ for all } z \text{ such that } |z|<R.$$

Now by Taylor's Theorem all the coefficients $c_n=0$, as desired.

Second possibility: $r<|z_o|$

Let us imagine drawing the line segment from z_o to zero. The point Z_1 is the point that lies on this segment, and on the circle $\{z\in\mathbb{C}:|z-z_o|=r\}$. Then $|Z_1|=|z_o|-r$. Here is a formula for Z_1:

$$Z_1=(|z_o|-r)\frac{z_o}{|z_o|}=\left(1-\frac{r}{|z_o|}\right)z_o$$

We note that the points z'_k along our line segment, at distances $\frac{k}{k+1}r$ from z_o, converge to Z_1, and that $h(z'_k)=0$. Once again we can use the argument we used on z_o, now applied to Z_1. Since $|Z_1|=|z_o|-r$, and $r=R-|z_o|$, we have $R-|Z_1|=R-(|z_o|-r)=2r$, and we know that $h(z)=0$ for all z such that $|z-Z_1|<2r$.

We note in passing that $|z - z_o| < r$ implies $|z - Z_1| \leq |z - z_o + z_o - Z_1| \leq |z - z_o| + |z_o - Z_1| = |z - z_o| + r < 2r$. That is, the first circle we obtained, in which $h(z) \equiv 0$, is included in the second one.

Now we can start over, with Z_1 playing the rôle of z_o. We now have $2r$ in place of r and Z_1 in place of z_o. We have to consider the two possibilities $2r \geq |Z_1|$ and $2r < |Z_1|$. If $2r \geq |Z_1|$, we only need to apply the argument once more, after which we can conclude that $h(z) \equiv 0$ for all $|z| < R$. Otherwise, we may have to repeat the process. Let's think of beginning with $m = 1$ and $r_m = r_1 = 2r$ in this case. So now $r_m < |Z_m|$.

Since $|Z_{m+1}| = |Z_m| - r_m < |Z_m|$ and $r_m = R - |Z_m|$, we have $R - |Z_{m+1}| = R - (|Z_m| - r_m) = 2r_m$. Thus $r_m = 2^m r$ and $|Z_{m+1}| = |z_o| - \sum_{j=0}^{m} 2^j r = |z_o| - (2^{m+1} - 1)r = |z_o| + r - 2^{m+1}r$. We know that $h(z) = 0$ for all z such that $|z - Z_m| < r_m$.

To construct the point Z_{m+1} on the segment from Z_m to zero (part of the segment from z_o to zero) we put

$$Z_{m+1} = (|Z_m| - r_m)\frac{Z_m}{|Z_m|} = \left(1 - \frac{r_m}{|Z_m|}\right) Z_m.$$

We keep constructing new points Z_{m+1} as long as $2^m r < |Z_m|$ (please be sure you agree!). Eventually, $2^m r \geq |Z_m|$ (Why?). Then we can apply the First Possibility argument, which finally shows that $h(z) = 0$ for all z such that $|z| < R$.

This completes the proof of the Theorem.

A Theorem on the product of power series

Another application of the Absolute Convergence Theorem for double series with complex terms:

Theorem: *Suppose*

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) := \sum_{n=0}^{\infty} b_n z^n$$

converge for $|z| < R$. *Then*

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{where} \quad c_n = \sum_{r=0}^{n} a_{n-r} b_r; \quad \text{the power series converges for } |z| < R.$$

Proof: The way to prove this is to start with the series on the right-hand side and arrive at the product later. We will make a change in the order of summation without justification, and obtain the justification at the end of the argument, a string of calculations:

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n z^n &= \sum_{n=0}^{\infty} \sum_{r=0}^{n} a_{n-r} b_r z^n \\
&= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} [r \leq n] a_{n-r} z^{n-r} b_r z^n \\
&= \sum_{r=0}^{\infty} b_r z^n \sum_{n=0}^{\infty} [r \leq n] a_{n-r} z^{n-r} \\
&= \sum_{r=0}^{\infty} b_r z^n \sum_{n=r}^{\infty} a_{n-r} z^{n-r} \\
&= \sum_{r=0}^{\infty} b_r z^n \sum_{n=0}^{\infty} a_n z^n = g(z)f(z) = f(z)g(z).
\end{aligned}$$

Since each series in the product in the last line converges absolutely for $|z| < R$, the double series can be summed in either order, so the steps we made are valid. They are actually reversible, but it would be hard to think of going from the last line to the one above it!