Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [15] State the *Completeness Axiom* [6]. Suppose that $\{x_n\}$ is a strictly increasing sequence that is bounded above. Use the Completeness Axiom to prove that $\lim_{n\to+\infty} x_n$ exists. In terms of the Completeness Axiom, what is $\lim_{n\to+\infty} x_n$? [9].

For all $S \subseteq \mathbb{R}$, if S is non-empty and bounded above, then there exists a unique element of $\mathbb{R}$, denoted $\sup(S)$, such that

(i) $\sup(S)$ is an upper bound for S, and

(ii) for all x in $\mathbb{R}$, if $x < \sup(S)$ then there exists $s \in S$ such that $x < s$.

Let S be the set of sequence terms. $S \neq \emptyset$ since $x_1 \in S$. S is bounded above by hypothesis. Thus $\sigma := \sup S$ exists in $\mathbb{R}$. Let $\epsilon > 0$. Then $\sigma - \epsilon < \sigma$ so there exists $N \in \mathbb{N}$ such that $x_N > \sigma - \epsilon$. Therefore for $n \geq N$,

$$\sigma - \epsilon < x_n \leq \sigma < \sigma + \epsilon. \text{ This is the definition of: } \lim_{n\to\infty x_n} = \sup S.$$

(2) [14] Define $\displaystyle\lim_{\substack{x\to x_o\\ x\in D}} f(x)$ [5]. Prove that if $\displaystyle\lim_{\substack{x\to x_o\\ x\in D}} f(x) = L$ and $\displaystyle\lim_{\substack{x\to x_o\\ x\in D}} g(x) = M$ then

$\displaystyle\lim_{\substack{x\to x_o\\ x\in D}} (f(x) + g(x)) = L + M$ [9].

There exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $x \in D$ and $0 < |x - x_o| < \delta$, then $|f(x) - L| < \epsilon$.

Let $\epsilon > 0$ be given. All x will be in D. Then there exists $\delta_1 > 0$ such that $0 < |x - x_o| < \delta_1 \Rightarrow |f(x) - L| < \epsilon/2$ and there exists $\delta_2 > 0$ such that $0 < |x - x_o| < \delta_2 \Rightarrow |g(x) - M| < \epsilon/2$. Let $\delta := \min\{\delta_1, \delta_2\}$. Then if $0 < |x - x_o| < \delta$,

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon. \text{ This is the definition of what was to be shown}$$

(3) [13] Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$. Prove that your answer is indeed the limit. You may use an ϵ–δ argument or Theorems and Examples done in class (suitably identified, with hypotheses checked).

Soln 1: $\frac{x^2-1}{x-1} = x+1 \to 2$ since the limit of a sum is the sum of the limits, and we noted in class that $\lim_{x\to x_o} x = x_o$ and $\lim_{x\to x_o} 1 = 1$.

Soln 2: $\frac{x^2-1}{x-1} = x + 1 \to 2$. Let $\epsilon > 0$ be given. Then $|x + 1 - 2| = |x - 1| < \epsilon$ if $|x - 1| < \delta := \epsilon$.

(4) [18] Prove (using axioms for a field!) that if F is a field, then
(a) $(-1)^2 = 1$ [5];
(b) If $x \in F$ and $x \neq 0$ then $x^2 \neq 0$ [5];
(c) If in addition F is an ordered field, prove that if $x \in F$ and $x \neq 0$ then $x^2 > 0$ [8].
Justify each step. Use the axiom number or (I prefer) the name of the axiom, such as "Add Comm, Distrib," etc. You may use without proof the results in Question 4 on Quiz 1.

(a) $(-1)^2 = (-1)(-1) = -(-1)$ by Q1#4; then $0 = -1 + (-(-1)) = (-(-1)) + (-1)$ so by uniqueness of additive inverses, $-(-1) = 1$, so $(-1)^2 = 1$.
(b) Suppose not. Then $x^2 = 0$. Let $y = 1/x$ (exists by mult-inv axiom). Then (strarting off using Q1#4) $0 = y^2 \cdot 0 = y^2x^2$ (subst) $= y(yx^2) = y((yx)x) = y(1x) = yx = 1$ (assoc, assoc, inv, unit, inv, contra $[1 \neq 0]$).

(c) Since $x \neq 0$, $x > 0$ or $x < 0$. If $x > 0$, $x^2 > 0$ (prod of pos pos). If $x < 0$ then $-x > 0$ (trichotomy and $x \neq 0$) so $(-x)^2 > 0$. That is,

$$0 < (-x)^2 = (-x)(-x) = ((-1)x)((-1)x) = (-1)(-1)x^2 = x^2 \text{ by (a) and unit axiom}$$

The other steps (some assoc & comm steps skipped): def, Q1#4 & subst, rearrangement(the assoc & comm steps skipped) $(-1)(-1) = (-1)^2$ (def).