Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [14] Prove that $\lim_{n\to\infty} x_n = 0 \iff \lim_{n\to\infty} |x_n| = 0$ [7]. Does this hold if the zero on the right is replaced by a non-zero number [2]? Why [2]? Apply the result to prove that $(-1)^n/n \to 0$ as $n \to \infty$ [3].

$\lim_{n\to\infty} x_n = 0$ means $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq N \Rightarrow |x_n - 0| < \epsilon$. Since $|x_n - 0| = |x_n|$ and $||x_n| - 0| = |x_n|$ as well, our statement "$x_n \to 0$" is actually the same as "$|x_n| \to 0$." No, this does not hold if the zero on the right is replaced by a non-zero number. Example: let $x_n = -2$. Then neither statement implies the other! We apply the "absolut-value result" by noting that $|(-1)^n/n| = 1/n \to 0$, as shown in class.

(2) [15] Prove that $\dfrac{n^2}{2^n} \to 0$ as $n \to \infty$. You might want to prove separately (as a Lemma) that for all n, $2^n > n^3/8$.

By writing down $n^3/2^n$ for n running from 1 thru 5 we see that the values go up and then down. Having in mind the *idea* that $n^3/2^n \to 0$, we can hope that they decrease for $n \geq 4$, so all of them are less than $4^3/2^4 = 64/16 = 4 < 8$. We use the "ratio" method: $\frac{(n+1)^3/2^{n+1}}{n^3/2^n} = (1+\frac{1}{n})^3/2 \leq 1$ if and only if $(1+\frac{1}{n})^3 \leq 2$ if and only if $(3/n) + (3/n^2) + (1/n^3) \leq 1$. All 3 quantities on the left decrease, and when $n = 4$ they add up to $\frac{3}{4} + \frac{3}{16} + \frac{1}{64} = \frac{3}{4} + \frac{1}{4} - \frac{1}{16} + \frac{1}{64} < 1$. Thus the sequence $\{n^3/2^n\}$ decreases strictly. It is thus bounded as stated, so $n^2/2^n < 8/n \to 0$ (shown in class), by the Lemma.

(3) [15] State the Bolzano-Weierstrass Theorem. Prove that if $\{x_n\}$ is a sequence that has a bounded subsequence then $\{x_n\}$ has convergent subsequence.

Every bounded sequence of real numbers has a bounded subsequence.

If $\{x_n\}$ has a bounded subsequence $\{x_{n_j}\}$ then there is a subsequence $\{x_{n_{j_\ell}}\}$ that converges, by *B–W*. But $\ell \mapsto x_{n_{j_\ell}}$ is a subsequence of $\{x_n\}$. (Remark: $j \mapsto n_j \uparrow$ strictly and $\ell \mapsto j_\ell \uparrow$ strictly. You should check that the composition of strictly increasing functions is a strictly increasing function!)

(4) [16] State the Nested Intervals Theorem [6]. Give an outline of the proof. Your outline should cover all the major points in the proof, and should not include any minor details [12]. Plan first, then write!

If $\{I_n\}$ is a nested sequence of closed and bounded intervals and $|I_n|(= b_n - a_n) \to 0$ then there exists exactly one x_o that is in every I_n.

By nestedness, the a_n increase and the b_n decrease, but with $a_n \leq b_n$. Then the a_n are bounded above by b_1, the b_n bounded below by a_1, so by the Axiom of Continuity (for us a Theorem!) both sequences converge. Their difference converges to zero (given), so they converge to the same limit, call it x_o. The Squeeze Theorem can then be used to show that if x' is in every I_n then $x' = x_o$.