Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [14] State [5] the *Boundedness Theorem.* Use the Boundedness Theorem to prove the *Extreme-value Theorem.* Just prove one of its two parts.

If $f : [a,\, b] \to \mathbb{R}$ is continuous then f is bounded.

Suppose no $f(x)$ is a maximum. By the Boundedness Theorem $\{f(x) : x \in [a,\, b]\}$ is bounded, and contains $f(a)$ so $M := \sup_{x\in[a,\, b]} f(x)$ exists. Then $f(x) < M$ for all $x \in [a,\, b]$ so $0 < \dfrac{1}{M - f(x)}$ is continuous on $[a,\, b]$ hence bounded by the Boundedness Theorem. With $\bar{M} := \sup_{x\in[a,\, b]} \frac{1}{M-f(x)} > 0$ we have, for all $x \in [a,\, b]$

$$\frac{1}{M - f(x)} \le \bar{M} \quad \text{or} \quad \frac{1}{\bar{M}} \le M - f(x) \quad \text{or} \quad f(x) \le M - \frac{1}{\bar{M}}, \quad \text{an upper bound less than the sup.}$$

This contradiction completes the proof.

(2) [15] Define *Cauchy sequence.* Prove that a Cauchy sequence that has a convergent subsequence converges.

A sequence $\{x_n\}$ is *Cauchy* if for all $\epsilon > 0$ there exists N such that $m \ge N$ and $n \ge N$ implies $|x_m - x_n| < \epsilon$.

Suppose x_{n_j} converges. Then there exists L such that for all $\epsilon > 0$ there exists J such that $j \ge J \Rightarrow |x_{n_j} - L| < \epsilon/2$. Also there exists N such that $m \ge N$ and $n \ge N$ implies $|x_m - x_n| < \epsilon/2$. Let $M := \max\{J,\, N\}$. We recall that $M \ge J \Rightarrow |x_{n_M} - L| < \epsilon/2$. Then for $n \ge M$ we also have $n_M \ge M$ so

$$|x_n - L| \le |x_n - x_{n_M}| + |x_{n_M} - L| < (\epsilon/2) + (\epsilon/2) = \epsilon, \quad \text{so } x_n \to L.$$

(3) [15] State the *Heine-Borel Theorem.* Suppose that $f : [a,\, b] \to \mathbb{R}$ is continuous. Then for each $x \in [a,\, b]$ there is $\delta_x > 0$ such that $|f(y) - f(x)| < 1$ if $y \in [a,\, b] \cap (x - \delta_x,\, x + \delta_x)$. Given: the intervals $(x - \delta_x,\, x + \delta_x)$ cover $[a,\, b]$. Use this and the Heine-Borel Theorem to prove that f is bounded on $[a,\, b]$.

If a family of open intervals covers a bounded closed interval then a finite subfamily exists that covers the interval.

From what was given we know there are finitely many points $x_1,\, \ldots,\, x_N$ such that the intervals $(x_i - \delta_{x_i},\, x_i + \delta_{x_i})$, $1 \le i \le N$, cover $[a,\, b]$. Thus for all $x \in [a,\, b]$, x is in some $(x_i - \delta_{x_i},\, x_i + \delta_{x_i})$
so $|f(x)| \le |f(x) - f(x_i)| + |f(x_i)| < 1 + \max\{|f(x_1)|,\, \ldots,\, |f(x_N)|\}$.

(4) [16] State the *Cauchy Criterion* [6]. Prove that the sequence $\{x_n\}$ defined by $x_n := \displaystyle\sum_{m=1}^{n} \frac{(-1)^m}{m}$ converges.

A sequence converges if and only if it is a Cauchy sequence.

Assume without loss that $n > m$. Then

$$|x_n - x_m| = \left| \sum_{j=m+1}^{n} \frac{(-1)^j}{j} \right| = \frac{1}{m+1} - \frac{1}{m+2} + - \cdots \pm \frac{1}{n}.$$

When we subtract, we decrease what was there last. When we add back, we add less than what we last subtracted. Thus we stay below the first term here, $\frac{1}{m+1}$. Thus if $\epsilon > 0$ and $N + 1 > 1/\epsilon$, $m \ge N$ and $n \ge N$ implies $|x_m - x_n| \le \frac{1}{m+1} < \epsilon$. Thus $\{x_n\}$ is Cauchy and hence converges.