

Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [15] Define *uniform continuity* of  $f : D \rightarrow \mathbb{R}$ , where  $D \subseteq \mathbb{R}$  is the domain of  $f$ . Prove that  $f(x) := \sqrt{x}$  is uniformly continuous on  $[1, \infty)$ . You might find that the Mean Value Theorem is useful here!

$f : D \rightarrow \mathbb{R}$  is unif. cont. if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $x \in D$  and  $y \in D$  and  $|x - y| < \delta$  then  $|f(x) - f(y)| < \epsilon$ .

If  $1 \leq x < y$  then  $\sqrt{y} - \sqrt{x} = (y - x)^{1/2} / \sqrt{c}$ , where  $x < c < y$ , by *MVT*. Since  $c > 1$ ,  $\sqrt{y} - \sqrt{x} < \frac{(y-x)}{2}$ , so given  $\epsilon > 0$  let  $\delta = 2\epsilon$ . Another way, without *MVT*, is to multiply and divide by  $\sqrt{y} + \sqrt{x}$ , which winds up in the denominator and is bigger than 2, with numerator  $y - x$ . Again we get that  $\delta = 2\epsilon$  works.

(2) [15] Prove that  $f(x) := \sqrt{x}$  is uniformly continuous on  $[0, 1]$ . Use this and the result from # 1 to prove that  $f(x) := \sqrt{x}$  is uniformly continuous on  $[0, \infty)$ .

$\sqrt{x}$  is unif. cont. on  $[0, 1]$  because it's cont. and  $[0, 1]$  is closed and bounded.

Let  $\epsilon > 0$  be given. Then there is  $\delta_0 > 0$  that works on  $[0, 1]$  for  $\epsilon/2$ , and there is  $\delta_1 > 0$  that works on  $[1, \infty)$  for  $\epsilon/2$ . Suppose that  $x$  and  $y$  are in  $[0, \infty)$  and  $|x - y| < \delta := \min\{\delta_0, \delta_1\}$ . If  $x$  and  $y$  are both in  $[0, 1]$ ,  $|\sqrt{x} - \sqrt{y}| < \epsilon/2 < \epsilon$ . If  $x$  and  $y$  are both in  $[1, \infty)$ ,  $|\sqrt{x} - \sqrt{y}| < \epsilon/2 < \epsilon$ . If (WLOG)  $x < 1 < y$ , then  $|x - 1| < \delta$  and  $|y - 1| < \delta$  so  $|f(x) - f(y)| \leq |f(x) - f(1)| + |f(1) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$ .

(3) [15] State the Mean Value Theorem. Verify that the Mean Value Theorem does not hold for the function  $f(x) = |x|$  on the interval  $[-1, 2]$ , and explain why.

If  $f$  is cont. on  $[a, b]$  and differentiable on  $(a, b)$  then there exists  $c \in (a, b)$  such that  $\frac{f(b) - f(a)}{b - a} = f'(c)$ .

Suppose *MVT* does hold for this function. Then for some  $c$ ,  $f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = 1/3$ . But  $f'(x) = \pm 1$  where it exists, so *MVT* does not hold. As to why it does not, we note that  $f'(0)$  does not exist, because the difference quotient  $\frac{|x| - |0|}{x - 0}$  is  $-1$  when  $x < 0$  and is  $+1$  when  $x > 0$ , so the one-sided limits,  $-1$  and  $+1$ , differ hence the limit of the difference quotient does not exist, and this means that  $f'(0)$  does not exist. This violates the hypothesis that  $f$  be differentiable at every point of  $(-1, 2)$ .

(4) [15] Suppose that  $f$  is differentiable on  $(a, b)$ ,  $a < u < v < b$ ,  $f'(u) > 0$  and  $f'(v) < 0$ . Prove that there exists  $c$ ,  $u < c < v$ , such that  $f'(c) = 0$ . Draw! Begin by convincing your self that the result is true. For substantial partial credit, write down your idea for a proof.

Since  $f'(u) > 0$ , there is an  $x$ ,  $u < x < v$  and close to  $u$  such that  $f(x) > f(u)$ . Since  $f'(v) < 0$ , there is a  $y$ ,  $u < y < v$  and close to  $v$  such that  $f(y) > f(v)$ . Thus  $\max\{f(x), f(y)\} > \max\{f(u), f(v)\}$  so there is a maximum value of  $f$  on  $[u, v]$  at some  $c \in (u, v)$ . We know that therefore  $f'(c) = 0$ . #4 is a Theorem, known as "The Intermediate Value Property For Derivatives." A function can be differentiable in an interval without the derivative being continuous. Nevertheless the derivative has the The Intermediate Value Property.