Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [15] Define *differentiability of* $f : D \to \mathbb{R}$, *at* x_o, where $D \subseteq \mathbb{R}$ is the domain of f and $x_o \in D$. Prove that if f is differentiable on $(0, 1]$ and continuous on $[0, 1]$ and $f'(x) \to L$ as $x \downarrow 0$ then f is differentiable at 0 (relative to $[0, 1]$) and $f'(0) = L$.

f is differentiable at x_o if $\displaystyle\lim_{\substack{x \to x_0 \\ x \in D}} \frac{f(x) - f(x_o)}{x - x_o}$ exists.

Here $x_o = 0$ and $D = [0, 1]$. Thus $\dfrac{f(x) - f(x_o)}{x - x_o} = \dfrac{f(x) - f(0)}{x} \to L$ by l'Hospital because we are in the $0/0$ case with $F(x) = f(x) - f(0)$ and $G(x) = x$. The derivatives exist on $(0, 1)$ and the functions are continuous on $[0, 1]$. And $F'(x)/G'(x) = f'(x) \to L$. We can also use MVT: f' exists on $(0, 1)$ and f is continuous on $[0, 1]$. Thus $\frac{f(x)-f(0)}{x} = f'(c)$, where $0 < c < x$. As $x \downarrow 0$, $c \to 0$, so $\frac{f(x)-f(0)}{x} = f'(c) \to L$ as $x \to 0$.

(2) [15] Prove that $f(x) := \sqrt{x}$ is differentiable on $(0, 1]$, continuous on $[0, 1]$ and not differentiable at 0 [12]. Find a point $c \in (0, 1)$ that satisfies the Mean Value Theorem for f [3]. Be specific about identifying Theorems proved in class! Stating Theorems you use is safest! "By a Theorem proved in class" will not be enough to earn points. You can also work directly.

If $0 < x_o \le 1$ we have $|\sqrt{x} - \sqrt{x_o}| = \frac{|x-x_o|}{\sqrt{x}+\sqrt{x_o}} \le \frac{|x-x_o|}{\sqrt{x_o}} < \epsilon$ if $|x - x_o| < \epsilon\sqrt{x_o}$. This gives continuity on $(0, 1]$.

If $0 < x_o \le 1$ we have $\frac{\sqrt{x}-\sqrt{x_o}}{x-x_o} = \frac{1}{\sqrt{x}+\sqrt{x_o}} \to \frac{1}{2\sqrt{x_o}}$. This gives differentiability on $(0, 1]$.

For continuity at $x_o = 0$, we let $\delta = \epsilon^2$ because then $|\sqrt{x} - \sqrt{x_o}| = \sqrt{x} < \epsilon$ if $0 < x < \epsilon^2$.

The difference quotient at $x_o = 0$ is $\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \to +\infty$ as $x \to 0$.

(3) [15] State both forms of the Fundamental Theorem of Calculus. Which form do we use to evaluate $\int_0^x t^n\,dt$? What is the value of that integral when $n \in \mathbb{N}$?

If f is continuous on $[a, b]$ then $F(x) := \int_a^x f(t)\,dt$ is differentiable on $[a, b]$ and $F'(x) = f(x)$ (relative to $D = [a, b]$).

If $F'(x) = f(x)$ on $[a, b]$ then $F(b) - F(a) = \int_a^b f(t)\,dt$. This is the version we use for $\int_0^x t^n\,dt = \frac{x^{n+1}}{n+1}$ when $n \in \mathbb{N}$.

(4) [15] State the Inverse Function Differentiation Theorem. **Use the Theorem** to find $\frac{d}{dy}\exp(y)$ in terms of $\log x$ or its inverse.

If f is strictly increasing on I and $x_o \in I$ and $f'(x_o) \ne 0$ then the inverse function $g(y)$ is differentiable at $y_o = f(x_o)$ and $g'(y_o) = \frac{1}{f'(x_o)}$.

Here g is the inverse of $\log x$. Thus $g'(y) = \frac{1}{f'(x)} = \frac{1}{1/x} = x = g(y)$.