Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [15] Define *differentiability of* $f : D \to \mathbb{R}^M$, *at* x_o, where $D \subseteq \mathbb{R}^N$ is the domain of f and $x_o \in D$. Prove that if f is differentiable at x_o then f is continuous at x_o.

f is differentiable at x_o if there exists an $M \times N$ matrix Λ such that $f(x_o + h) = f(x_o) + \Lambda h + o(h)$, where $o(h)/\|h\| \to 0$ as $\|h\| \to 0$.

Since $f(x) \to f(x_o)$ means continuity and $x = x_o + h \to x_o$ as $\|h\| \to 0$,
$f(x_o + h) - f(x_o) = \Lambda h (\to 0 \text{ as } \|h\| \to 0) + o(h)(\to 0 \text{ as } \|h\| \to 0) \to 0$ as $\|h\| \to 0$.

(2) [15] Let $f(x, y) := \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & \text{if } x^2 + y^2 > 0, \\ 0, & \text{if } x^2 + y^2 = 0. \end{cases}$ Determine whether or not f is continuous at $(0, 0)$. Justify your answer! Calculate the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0, 0)$.

f is *not* continuous at $(0, 0)$. We let $x = y^2$. Then away from the origin $f(y^2, y) = y^4/2y^4 = 1/2 \not\to 0$.

To compute the partial derivatives use the definition (because $(0, 0)$ is an "exceptional" point in the definition of f). At $(0, 0)$ $\dfrac{\partial f}{\partial x} = \lim_{h \to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \dfrac{h \cdot 0^2}{h^2 + 0^4} \Big/ h = 0/h = 0 \to 0$. Similarly $\frac{\partial f}{\partial y} = 0$ at $(0, 0)$.

(3) [15] Define *open set, closed set, limit point.* Give an example of a set that is neither open nor closed. What is the set of limit points of the set of all rational numbers in $(0, 1)$? Justify your answers!

A set A is open if every $x \in A$ is contained in an open ball contained in A.
A set A is closed if A contains all its limit points.
A point p_o is a limit point "of" a set A if for all $\epsilon > 0$ there exists $p \in A$ such that $d(p, p_o) < \epsilon$ and $p \neq p_o$.
The set $A := (0, 1]$ is not closed because 0 is a limit point of A that is not in A and there is no ball (open interval here) contained in A that contains 1, a point of A. The set of limit points of the set of all rational numbers in $(0, 1)$ is $[0, 1]$ because every real number is the limit of a sequence of rationals; but no real number *outside* $[0, 1]$ is a limit point of $[0, 1]$, hence not a limit point of any subset of $[0, 1]$!

(4) [15] Define *metric* and *metric space.* Let ℓ^∞ denote the collection of all bounded sequences $x = \{x_n\}$ of real numbers. Prove that $d(x, y) := \sup |x_n - y_n|$ is a metric on ℓ^∞.

$d : X \times X \to \mathbb{R}$ is a metric on X if
(1) $d(x, y) \geq 0$ for all $x, y \in X$;
(2) $d(x, y) = 0 \Rightarrow x = y$, for all $x, y \in X$;
(3) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A metric space is a set X together with a metric on X. This can be expressed also as: a pair (X, d), where d is a metric on X.

$d(x, y) = \sup_n |x_n - y_n| \geq 0$;
$d(x, y) = \sup_n |x_n - y_n| = 0 \Rightarrow |x_n - y_n| = 0$ for all $n \in \mathbb{N} \Rightarrow x_n = y_n$ for all $n \in \mathbb{N} \Rightarrow x = y$;
$d(x, y) = \sup_n |x_n - y_n| = \sup_n |y_n - x_n| = d(y, x)$;
$d(x, y) = \sup_n |x_n - y_n| = \sup_n |x_n - z_n + z_n - y_n| \leq \sup_n |x_n - z_n| + |z_n - y_n| \leq \sup_n |x_n - z_n| + \sup_n |z_n - y_n|$
$= d(x, y) + d(y, z)$.