

§2 Some consequences of the mere existence of Riemann-Stieltjes integrals

Integration by parts

Wheeden & Zygmund show, in Theorem (2.21) of *Measure and Integral*, that if the Riemann-Stieltjes integral $\int_a^b f d\alpha$ exists, this integration-by-parts formula holds:

$$(2.1) \quad \int_a^b f d\alpha = - \int_a^b \alpha df + f(b)\alpha(b) - f(a)\alpha(a).$$

In the applications I have far, far back in my mind, the integral on the right would have to be $\int_a^b df \alpha$, in order to keep the order of “multiplication” the same. Here, we deal with real-valued or complex-valued functions of one variable only. The proof of (2.1) amounts to rearranging the Riemann-Stieltjes sums, adding and subtracting terms in such a way that the ξ_i become partition points and the x_i become selection-vector components when $1 < i < n_\pi$. There are some leftovers, and these can be modified, to produce the “boundary” term, $f(x)\alpha(x)|_a^b$, or $f\alpha|_a^b$.

Proof: We are given that the integral on the left in (2.1) exists. Hence, given $\epsilon > 0$ there exists $\delta > 0$ such that for every partition π of $[a, b]$ with $\text{mesh}(\pi) < \delta$ and every selection vector ξ compatible with π ,

$$|RS(f, \alpha, \pi, \xi) - \int_a^b f d\alpha| < \epsilon.$$

Using this we have to show that for all $\epsilon > 0$ there exists $\delta' > 0$ such that for every partition π' of $[a, b]$ with $\text{mesh}(\pi') < \delta'$ and every selection vector ξ' compatible with π' ,

$$\left| RS(\alpha, f, \pi', \xi') - \left(- \int_a^b \alpha df + f(x)\alpha(x)|_a^b \right) \right| < \epsilon.$$

Given $\epsilon > 0$ let us put $\delta' := \delta/2$ and suppose that $\text{mesh}(\pi') < \delta'$. Then (please check each step!)

$$RS(\alpha, f, \pi', \xi') = \sum_{j=1}^{n_{\pi'}} \alpha(\xi'_j)(f(x'_j) - f(x'_{j-1})) = \sum_{j=1}^{n_{\pi'}} \alpha(\xi'_j)f(x'_j) - \sum_{j=1}^{n_{\pi'}} \alpha(\xi'_j)f(x'_{j-1}) \tag{2.2a}$$

$$= \sum_{j=1}^{n_{\pi'}} \alpha(\xi'_j)f(x'_j) - \sum_{j=0}^{n_{\pi'}-1} \alpha(\xi'_{j+1})f(x'_j) \quad (\text{we made } j \mapsto j+1 \text{ in the second sum}) \tag{2.2b}$$

$$= \sum_{j=1}^{n_{\pi'}-1} (\alpha(\xi'_j) - \alpha(\xi'_{j+1}))f(x'_j) + \alpha(\xi'_{n_{\pi'}})f(x'_{n_{\pi'}}) - \alpha(\xi'_1)f(x'_0) \tag{2.2c}$$

$$(2.2) \quad = \sum_{j=1}^{n_{\pi'}-1} (\alpha(\xi'_j) - \alpha(\xi'_{j+1}))f(x'_j) + \alpha(\xi'_{n_{\pi'}})f(b) - \alpha(\xi'_1)f(a) \quad (\text{Why?}) \tag{2.2d}$$

$$= \sum_{j=1}^{n_{\pi'}-1} (\alpha(\xi'_j) - \alpha(\xi'_{j+1}))f(x'_j) - (\alpha(b) - \alpha(\xi'_{n_{\pi'}}))f(b) - (\alpha(\xi'_1) - \alpha(a))f(a) + \alpha(b)f(b) - \alpha(a)f(a) \tag{2.2e}$$

$$= - \sum_{j=1}^{n_{\pi'}-1} (\alpha(\xi'_{j+1}) - \alpha(\xi'_j))f(x'_j) - (\alpha(b) - \alpha(\xi'_{n_{\pi'}}))f(b) - (\alpha(\xi'_1) - \alpha(a))f(a) + \alpha(b)f(b) - \alpha(a)f(a). \tag{2.2f}$$

All this rearranging has aimed toward seeing the sum $RS(\alpha, f, \pi', \xi')$ that we started with as a sum $RS(f, \alpha, \pi, \xi)$ plus a “boundary” term. Let’s observe that every selection vector ξ has the property that $a \leq \xi_{i-1} \leq \xi_i \leq b$. To make it easy at first let’s suppose that for our selection vector ξ' we have $x'_{j-1} < \xi'_j < x'_j$ for $1 \leq j \leq n_{\pi'}$. Then by setting $\xi'_0 := a$ and $\xi'_{n_{\pi'}+1} := b$ we have $a = \xi'_0 < \xi'_1 < \dots < \xi'_{n_{\pi'}} < \xi'_{n_{\pi'}+1} = b$, and this is a partition of $[a, b]$ that we can call π . For us, π will have points $x_i := \xi'_i$, $0 \leq i \leq n_{\pi'} + 1 =: n_\pi$. This is our “restricted” case.

To continue our quest to see $RS(\alpha, f, \pi', \xi')$ as a sum $RS(f, \alpha, \pi, \xi)$ plus a “boundary” term, we need a selection vector compatible with $\pi (= \xi')$. We can take $\xi_1 := a$, for that choice makes the term $(\alpha(\xi'_1) - \alpha(a))f(a)$ (in (2.2)f)

take on the form $f(\xi_1)(\alpha(x_1) - \alpha(x_0))$. The other term that's not under the summation sign, $(\alpha(b) - \alpha(\xi'_{n_{\pi'}}))f(b)$, takes on the form $f(\xi_{n_{\pi}})(\alpha(x_{n_{\pi}}) - \alpha(x_{n_{\pi}-1}))$ if we make $\xi_{n_{\pi}} := b$. Reminder: $n_{\pi} = n_{\pi'} + 1$. For $1 < i < n_{\pi}$, we put $\xi_i := x'_{i-1}$.

We can now rewrite (2.2) somewhat, with (2.2)f preserved, using our redos of the “lonely” terms:

$$\begin{aligned}
 RS(\alpha, f, \pi', \xi') &= - \sum_{j=1}^{n_{\pi'}-1} (\alpha(\xi'_{j+1}) - \alpha(\xi'_j))f(x'_j) - (\alpha(b) - \alpha(\xi'_{n_{\pi'}}))f(b) - (\alpha(\xi'_1) - \alpha(a))f(a) \\
 &\quad + \alpha(b)f(b) - \alpha(a)f(a) \quad (\text{and we'll change the index of summation too: } j \mapsto i - 1) \\
 &= - \sum_{i=2}^{n_{\pi'}} (\alpha(\xi'_i) - \alpha(\xi'_{i-1}))f(x'_{i-1}) - f(\xi_{n_{\pi}})(\alpha(x_{n_{\pi}}) - \alpha(x_{n_{\pi}-1})) - f(\xi_1)(\alpha(x_1) - \alpha(x_0)) \\
 &\quad + \alpha(b)f(b) - \alpha(a)f(a) \quad (\text{next, we'll bring the “lonely” terms into the sum}) \\
 (2.3) \quad &= - \sum_{i=1}^{n_{\pi'}+1} (\alpha(\xi'_i) - \alpha(\xi'_{i-1}))f(x'_{i-1}) + \alpha(b)f(b) - \alpha(a)f(a) \\
 &= - \sum_{i=1}^{n_{\pi}} (\alpha(x_i) - \alpha(x_{i-1}))f(\xi_i) + \alpha(b)f(b) - \alpha(a)f(a) \\
 &= - \sum_{i=1}^{n_{\pi}} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) + f\alpha|_a^b.
 \end{aligned}$$

To get to the very last line we changed the product order and wrote our boundary terms in a standard way. We would not make that change in the order of multiplication if we were not allowed to. The sum in the last line is a Riemann-Stieltjes sum because $x_0 = a$, $x_{n_{\pi}} = b$ and $x_i = \xi'_i$ (recall that we defined $\xi'_0 = a$ and $\xi'_{n_{\pi'}+1} = b$) and $x_{i-1} = \xi'_{i-1} < x'_{i-1} = \xi_i < \xi'_i = x_i$. Remember, we are still in the restricted case.

Now we can make our desired rewrite of (2.2):

$$\begin{aligned}
 (2.4) \quad RS(\alpha, f, \pi', \xi') &= - \sum_{i=1}^{n_{\pi}} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) + f(x)\alpha(x)|_a^b \\
 &= - RS(f, \alpha, \pi, \xi) + f(x)\alpha(x)|_a^b.
 \end{aligned}$$

What is $\text{mesh}(\pi)$? This is $\max_{1 \leq i \leq n_{\pi}} (x_i - x_{i-1}) = \max_{1 \leq j \leq n_{\pi'}+1} (\xi'_j - \xi'_{j-1})$. For each j , $\xi'_j - \xi'_{j-1} < x'_j - x'_{j-2}$ because ξ'_j can be very close to x'_j and ξ'_{j-1} can be very close to x'_{j-2} . Hence

$$\xi'_j - \xi'_{j-1} < (x'_j - x'_{j-1}) + (x'_{j-1} - x'_{j-2}) \leq 2\text{mesh}(\pi') < 2\delta' = \delta, \quad \text{so } \text{mesh}(\pi) = \text{mesh}(\xi') < \delta.$$

By hypothesis, $|-RS(f, \alpha, \pi, \xi) + \int_a^b f d\alpha| < \epsilon$. Therefore,

$$\left| RS(\alpha, f, \pi', \xi') - \left(- \int_a^b f d\alpha + f(x)\alpha(x)|_a^b \right) \right| < \epsilon.$$

We have (please double-check!) verified that the conditions for the existence of $\int_a^b \alpha df$ are true, where we used $-\int_a^b f d\alpha + f(x)\alpha(x)|_a^b$ as our “RSI” in the definition of $\int_a^b \alpha df$. Thus the integral exists and the “integration-by-parts” formula holds.

When we began the proof we restricted our attention to the special case when all the selection points (coordinates of ξ) lay in the interiors of their intervals of π . The restriction was made in the paragraph after (2.2). To remove the restriction we have to allow $\xi'_{j+1} = \xi'_j$ for some j . (Why can't we have $\xi'_{j+1} = \xi'_j = \xi'_{j-1}$ for some j ?) If we indeed have $\xi'_{j+1} = \xi'_j$ for some j , then the term in (2.2)f with $\xi'_{j+1} = \xi'_j$ will be zero. (Why can't we have $j = 0$

in this situation?) We simply remove that ξ'_j and the x'_j from our lists and decrease $n_{\pi'}$ by one. Note that we only remove *one* of the pair of duplicates (and the x'_j that is equal to them both). When all the duplicates have been removed, we will have distinct ξ 's left. But because no two consecutive ξ 's can be farther apart than *twice* mesh(π'), or $2 \cdot \delta/2$, the mesh of our new partition π is less than δ . The argument can thus continue as in the restricted case. This completes the proof.

Linearity properties of Riemann-Stieltjes integrals

(2.5) **Theorem:** $\int_a^b f d\alpha$ is linear in both f and α if all of the integrals involved exist.

Proof: This is actually an outline for a complete proof. We want to show that

$$(2.6) \quad \int_a^b (Af + Bg) d(C\alpha + D\beta) = AC \int_a^b f d\alpha + AD \int_a^b f d\beta + BC \int_a^b g d\alpha + BD \int_a^b g d\beta,$$

assuming that each of the integrals on the right exists. Let us number the integrals one through four, reading left-to-right, and number corresponding Riemann-Stieltjes sums and related quantities the same way.

The proof is completed by noticing first that

$$RS(Af + Bg, C\alpha + D\beta, [a, b], \pi, \xi) = AC \cdot RS(f, \alpha, [a, b], \pi, \xi) + AD \cdot RS(f, \beta, [a, b], \pi, \xi) + BC \cdot RS(g, \alpha, [a, b], \pi, \xi) + BD \cdot RS(g, \beta, [a, b], \pi, \xi).$$

We then apply the existence of each integral, obtaining for each one a mesh size $\delta_k > 0$ (where $k = 1, 2, 3$ or 4) that guarantees that the corresponding Riemann-Stieltjes sum (including its constant multiple) is closer to the corresponding integral (including its constant multiple) than $\epsilon/4$. Finally, we use a partition π whose mesh is smaller than the least of the δ_k to show that the integral on the left exists and that the formula is correct.

Riemann-Stieltjes integrals split into subinterval pieces

(2.7) **Theorem:** If $\int_a^b f d\alpha$ exists and $a < c < b$ then $\int_a^c f d\alpha$ and $\int_c^b f d\alpha$ exist and $\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$.

Proof: Let us show that $\int_a^c f d\alpha$ exists first. We will use the Cauchy Criterion. Thus, given $\epsilon > 0$ we seek $\delta > 0$ such that for all $\sigma|[a, c]$ and $\sigma'|[a, c]$, mesh(σ) $< \delta$ and mesh(σ') $< \delta \Rightarrow |RS(f, [a, c], \sigma) - RS(f, [a, c], \sigma')| < \epsilon$.

We now use the existence of $\int_a^b f d\alpha$. Using the same $\epsilon > 0$ we obtain (hypothesis!) $\delta > 0$ such that for all $\pi|[a, b]$ and $\pi'|[a, b]$, mesh(π) $< \delta$ and mesh(π') $< \delta \Rightarrow |RS(f, [a, b], \pi) - RS(f, [a, b], \pi')| < \epsilon$. Here, as in the last paragraph, compatible selection vectors are “understood.”

With all this done we choose a partition $\rho|[c, b]$ with mesh(ρ) $< \delta$. We choose a selection vector η compatible with ρ . Note that so far we are ignoring $[a, c]$!

Next we take arbitrary $\sigma|[a, c]$ and $\sigma'|[a, c]$, with mesh(σ) $< \delta$ and mesh(σ') $< \delta$ and arbitrary selection vectors ζ and ζ' compatible with σ and σ' respectively. We will now drag ρ and η in.

We put $\pi := \sigma \cup \rho$, $\pi' := \sigma' \cup \rho$. “By construction,” $\pi|[a, b]$ and $\pi'|[a, b]$ and mesh(π) $< \delta$ and mesh(π') $< \delta$. Next we form the selection vectors $\xi := (\zeta, \eta)$ and $\xi' := (\zeta', \eta)$ by listing the coordinates of ζ (or ζ') first, followed by the coordinates of η . Please check (in your head) that ξ and ξ' are compatible with π and π' respectively.

Then π and π' and ξ and ξ' meet the conditions set by our hypothesis that $\int_a^b f d\alpha$ exists. Therefore $|RS(f, [a, b], \pi, \xi) - RS(f, [a, b], \pi', \xi')| < \epsilon$. But

$$RS(f, [a, b], \pi, \xi) - RS(f, [a, b], \pi', \xi') = RS(f, [a, c], \sigma, \zeta) - RS(f, [a, c], \sigma', \zeta'),$$

because all the terms of the Riemann-Stieltjes sums that come from $[c, b]$ cancel each other. Therefore

$$\epsilon > |RS(f, [a, b], \pi, \xi) - RS(f, [a, b], \pi', \xi')| = |RS(f, [a, c], \sigma, \zeta) - RS(f, [a, c], \sigma', \zeta')|$$

and we have shown that $\int_a^c f d\alpha$ exists. The same argument can be used to show that $\int_c^b f d\alpha$ exists.

It remains to show that the formula (2.6) is correct. This is easy to do, *provided that* we remember the proof we have done so far!

We can let $\pi|_{[a, b]}$ be any partition of $[a, b]$ with $\text{mesh}(\pi) < \delta$, choose any selection vector ξ compatible with π , and know that $\left|RS(f, [a, b], \pi, \xi) - \int_a^b f d\alpha\right| \leq \epsilon$. (Why the \leq and not the $<$?)

If we wish, we can choose such a π that contains c . Then with $\sigma := \pi \cap [a, c]$, $\rho := \pi \cap [c, b]$ and with ξ written as (ζ, η) ,

$$\begin{aligned} \left| \int_a^b f d\alpha - \int_a^c f d\alpha - \int_c^b f d\alpha \right| &\leq \left| \int_a^b f d\alpha - RS(f, [a, b], \pi, \xi) \right| + \left| RS(f, [a, b], \pi, \xi) - \int_a^c f d\alpha - \int_c^b f d\alpha \right| \\ &\leq \epsilon + \left| RS(f, [a, c], \sigma, \zeta) + RS(f, [c, b], \rho, \eta) - \int_a^c f d\alpha - \int_c^b f d\alpha \right| \\ &\leq \epsilon + \left| RS(f, [a, c], \sigma, \zeta) - \int_a^c f d\alpha \right| + \left| RS(f, [c, b], \rho, \eta) - \int_c^b f d\alpha \right| \leq 3\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, the equation in (2.7) holds.

Caution! It is possible that both of $\int_a^c f d\alpha$ and $\int_c^b f d\alpha$ exist, but that $\int_a^b f d\alpha$ does not exist! We'll return to this point in the section on necessary conditions. In Section 3 we will cover the "common-discontinuity" phenomenon Wheeden and Zygmund discuss in Chapter 2, §3, remark 4 (between (2.15) and (2.16)). Then we can exhibit a counterexample where both of $\int_a^c f d\alpha$ and $\int_c^b f d\alpha$ exist, but $\int_a^b f d\alpha$ does not. See **An Example** in Section 3, just before (3.8)

What has been covered in this section applies to *all* Riemann-Stieltjes integrals, including those that follow! I repeat: in most of what has been said in this section, we have **assumed** that all the Riemann-Stieltjes integrals under discussion **exist**. This will come in Section 5. We want to have some large collections of functions f and integrators α for which $\int_a^b f d\alpha$ is known to exist. That continuity plays a rôle, especially for f , has already been mentioned. Our goal is to introduce a condition, *bounded variation*, in Section 4, that will play a major rôle for α .