§3 Some necessary conditions for the existence of $\int_a^b f\,d\alpha$

This section is devoted to proving two theorems with rather "technical" proofs, based on one observation inspired by the maxim "Study special cases."

An observation

The Cauchy criterion for the existence of Riemann-Stieltjes integrals puts conditions on the differences between two Riemann-Stieltjes sums for a function f with respect to an "integrator" α, on an interval $[a, b]$. When written in full detail,

$$RS(f, \alpha, [a, b], \pi, \xi) - RS(f, \alpha, [a, b], \pi', \xi') = \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - \sum_{i'=1}^{n_{\pi'}} f(\xi'_{i'})(\alpha(x'_{i'}) - \alpha(x'_{i'-1})).$$

If we look at the special case when $\pi' = \pi$ we get

$$RS(f, \alpha, [a, b], \pi, \xi) - RS(f, \alpha, [a, b], \pi, \xi') = \sum_{i=1}^{n_\pi} (f(\xi_i) - f(\xi'_i))(\alpha(x_i) - \alpha(x_{i-1})).$$

Finally, if we look at the very special case when exactly one of the coordinates in ξ' differs from the corresponding coordinate in ξ,

$$RS(f, \alpha, [a, b], \pi, \xi) - RS(f, \alpha, [a, b], \pi, \xi') = (f(\xi_P) - f(\xi'_P))(\alpha(x_P) - \alpha(x_{P-1})), \tag{3.1}$$

where P is the sole coordinate index i with $\xi'_i \neq \xi_i$. Thus a condition that must hold for every difference of Riemann-Stieltjes sums (that satisfy some side condition, perhaps) must hold for this very special case (provided it satisfies the same side condition).

We will prove two Theorems that say (i) for all $x \in [a, b]$, at least one of f and α must be continuous at x and (ii) each of f and α must be bounded on the set where the other "changes." In both Theorems we will show that there is $\epsilon > 0$ such that for all $\delta > 0$ we can construct a partition $\pi | [a, b]$ with $\operatorname{mesh}(\pi) < \delta$ and selection vectors ξ and ξ', whose coordinates differ at only one index, such that the difference in (3.1) has absolute value at least ϵ. This will contradict the Cauchy criterion for existence of the Riemann-Stieltjes integral in question.

The first theorem

We need a new definition describing discontinuities at a point: $h(x)$ has a *removable discontinuity at* $\bar{x}$ if $\lim_{x\to\bar{x}} h(x)$ exists but $\lim_{x\to\bar{x}} h(x) \neq h(\bar{x})$. Example: $h(x) = 0$ if $x \neq 0$ and $h(0) = 1$. But $h(x)$ has a *non-removable discontinuity at* $\bar{x}$ if $\lim_{x\to\bar{x}} h(x)$ does not exist (or, we say the discontinuity at $\bar{x}$ is not removable). Example: $h(x) = 0$ if $x < 0$ and $h(x) = 1$ if $x \geq 0$. Note that these examples have simple discontinuities. A discontinuity of the second kind is always non-removable. A removable discontinuity is always a simple discontinuity, but not every simple discontinuity is removable.

(3.2) **Theorem:** *If f and α have a discontinuity at the same point, then $\int_a^b f\,d\alpha$ does not exist.*

Proof: A special case, that a common discontinuity occurs at an endpoint, is fairly easy. We may assume that a common discontinuity occurs at $x = a$. Both functions fail to have a limit at $x = a$ that coincides with the function's value (perhaps one or both do have limits at $x = a$, but in both cases the limit (even if it exists) is not the same as the function's value at $x = a$). Thus there exists $\zeta > 0$ such that for all $\delta > 0$ there exists $y \in [a, b]$ such that $a < y < a + \delta$ and $|f(y) - f(a)| \geq \zeta$. Also, there exists $\eta > 0$ such that for all $\delta > 0$ there exists $w \in [a, b]$ such that $a < w < a + \delta$ and $|\alpha(w) - \alpha(a)| \geq \eta$. We now put $\epsilon := \zeta\eta$ and let $\delta > 0$ be given. Thus there exists $w \in [a, b]$ such that $a < w < a + \delta$ and $|\alpha(w) - \alpha(a)| \geq \eta$. Here is why "fairly easy," rather than "easy" was mentioned: we now set $\delta' := w - a < \delta$. Then there exists $y_1 \in [a, b]$ such that $a < y_1 < a + \delta' = w$ and $|f(y_1) - f(a)| \geq \zeta$.

We will construct our partition now. Variations on this method of construction will be used to construct the other partitions that we use in this section.

We choose a natural number N so large that $(b-a)/N < \delta$ and we let

$$\pi := \{a,\, w\} \cup \{w + (i-1)\frac{b-w}{N} : 2 \le i \le N+1\}. \tag{3.3}$$

Since $I_1 = [a,\, w]$ and $w - a < \delta$, and since the length of all the other intervals is $(b-w)/N < (b-a)/n < \delta$, $\operatorname{mesh}(\pi) < \delta$. Next we need to choose our selection vectors, ξ and ξ'. All but one of the components of both selection vectors will be the same. For $i > 1$ we set $\xi_i = w + (i-1)\frac{b-w}{N} = \xi'_i$. For $i = 1$ we put $\xi_1 = a$ and $\xi'_1 = y_1$, obtained before we constructed π. Then, from (3.1),

$$|RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \xi) - RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \xi')| = |f(y_1) - f(a)||\alpha(w) - \alpha(a)| \ge \epsilon.$$

This contradicts the Cauchy criterion for the existence of Riemann-Stieltjes integrals when $\epsilon/2$ is used in it, instead of the ϵ we defined.

Next we suppose that f and α are discontinuous at $\bar{x} \in (a,\, b)$. There are two cases: (I) α has a removable discontinuity at $\bar{x}$ and (II) the discontinuity of α at $\bar{x}$ is not removable.

Case (I). We will show that there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $\pi|[a,\, b]$ with $\operatorname{mesh}(\pi) < \delta$ and there exist selection vectors $\hat{\xi}$ and $\tilde{\xi}$, both compatible with π, such that

$$|RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \hat{\xi}) - RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \tilde{\xi})| \ge \epsilon.$$

Let $\delta > 0$ be given. Since α has a removable discontinuity at $\bar{x}$, there exists $\zeta > 0$ and there exists $t' \in [a,\, b]$ such that $\bar{x} - \delta < t' < \bar{x}$ but $|\alpha(t') - \alpha(\bar{x})| \ge \zeta$ and there exists $t'' \in [a,\, b]$ such that $\bar{x} < t'' < \bar{x} + \delta$ but $|\alpha(t'') - \alpha(\bar{x})| \ge \zeta$. We need this next: $\delta' := \min\{\bar{x} - t',\, t'' - \bar{x}\}$.

Since f is discontinuous at $\bar{x}$, there exists $\eta > 0$ and there exists $t''' \in [a,\, b]$ such that $0 < |t''' - \bar{x}| < \delta'$ but $|f(t''') - f(\bar{x})| \ge \eta$. We don't know the side of $\bar{x}$ that t''' lies on.

We will construct our partition now. We choose a natural number N so large that $(b-a)/N < \delta$ and we let

$$\pi := \{a + k\frac{t'-a}{N} : 0 \le k \le N\} \cup \{t',\, \bar{x},\, t''\} \cup \{t'' + \ell\frac{b-t''}{N} : 0 \le \ell \le N\}. \tag{3.4}$$

By construction, $\operatorname{mesh}(\pi) < \delta$. Next we need to choose our selection vectors, $\hat{\xi}$ and $\tilde{\xi}$. All but one of the components of both selection vectors will be the same. This means that all but one of the terms in $RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \hat{\xi}) - RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \tilde{\xi})$ will be zero. We can now do the details.

Three sets appear in the union in (3.4). Each of t' and t'' belong to two of them. If we think of π as having points x_i, then $\bar{x} = x_{N+1}$ and $n_\pi = 2N+2$. Also, $x_N = t'$ and $x_{N+2} = t''$.

If $t''' < \bar{x}$ we set $\hat{\xi}_{N+1} = t'''$ and we set $\tilde{\xi}_{N+1} = \bar{x}$. For all other i we set $\hat{\xi}_i = x_i = \tilde{\xi}_i$. Since

$$RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \hat{\xi}) - RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \tilde{\xi}) = \sum_{i=1}^{2N+2} (f(\hat{\xi}_i) - f(\tilde{\xi}_i))(\alpha(x_i) - \alpha(x_{i-1})), \tag{3.5}$$

and $(f(\hat{\xi}_i) - f(\tilde{\xi}_i)) = 0$ unless $i = N+1$. Thus

$$RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \hat{\xi}) - RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \tilde{\xi}) = (f(\hat{\xi}_{N+1}) - f(\tilde{\xi}_{N+1}))(\alpha(x_{N+1}) - \alpha(x_N)) = (f(t''') - f(\bar{x}))(\alpha(\bar{x}) - \alpha(t')).$$

Earlier, we found t''' and t' such that $|f(t''') - f(\bar{x})| \ge \eta$ and $|\alpha(t') - \alpha(\bar{x})| \ge \zeta$, where η and ζ were positive constants that depended only on the assumption that both functions were discontinuous at $\bar{x}$ and that the discontinuity of α is removable. We can put absolute values on the last equation and get

$$|RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \hat{\xi}) - RS(f,\, \alpha,\, [a,\, b],\, \pi,\, \tilde{\xi})| = |f(t''') - f(\bar{x})||\alpha(\bar{x}) - \alpha(t')| \ge \eta\zeta.$$

We now set $\epsilon := \eta\zeta$. Since $\delta > 0$ was arbitrary we have shown what we needed to complete Case (1), if $t''' < \bar{x}$. If, instead, $t''' > \bar{x}$, we set $\hat{\xi}_{N+2} = t'''$ and we set $\tilde{\xi}_{N+2} = \bar{x}$. For all other i we set $\hat{\xi}_i = x_i = \tilde{\xi}_i$. The rest of the argument is almost exactly the same. We just change each occurence of N that comes after (3.5) to $N + 1$. This completes the proof in Case (I).

Case (II). What we have to do is exactly the same as in Case (I). We have to go about showing it differently.

This time we assert that there exists $\zeta > 0$ such that for all $\delta > 0$ there exist $t' < \bar{x} < t''$, with $t' \in [a,\, b]$ and $t'' \in [a,\, b]$, with $t'' - t' < \delta$ and such that $|\alpha(t'') - \alpha(t')| \geq \zeta$ (we must verify our assertion). Then, just as before we know that there exists $\eta > 0$ such that for all $\delta > 0$ there exists $t''' \in (t',\, t'')$ such that $0 < |t''' - \bar{x}| < \delta$ and $|f(t''') - f(\bar{x})| \geq \eta$. This part requires no further verification. Next we modify the partition constructed in (3.4) by leaving out the $\bar{x}$ in the middle set. This changes the numbering slightly; now $n_\pi = 2N + 1$. But the idea of the argument can now be carried out. This time it is a little easier because we pay no attention to which side of $\bar{x}$ that t''' lies on.

Now we turn our attention to verifying that there exists $\zeta > 0$ such that for all $\delta > 0$ there exist $t' < \bar{x} < t''$, with $t' \in [a,\, b]$ and $t'' \in [a,\, b]$, with $t'' - t' < \delta$ and such that $|\alpha(t'') - \alpha(t')| \geq \zeta$, now assuming that the discontinuity of α at $\bar{x}$ is not removable. This is the hardest part of the proof. One thing that makes it hard is that it is easier to prove a generalized version of what we need than it is to prove what we asserted! The reason is that our present situation is needlessly complicated by the natural order of $\mathbb{R}$. We thus will state a lemma, explain it a little, use it for what we want, then prove the lemma.

(3.6) **Lemma:** *Let X and Y be metric spaces and suppose that Y is complete. Suppose that $E = E_1 \cup E_2 \subseteq X$ and that $\bar{x}$ is a limit point of both sets E_1 and E_2 but that $\bar{x}$ belongs to neither. Suppose that $f : E \to Y$. Then $\lim\limits_{x \to \bar{x},\, x \in E} f(x)$ exists if and only if*

$$\lim_{\substack{(x_1,\, x_2) \to (\bar{x},\, \bar{x}) \\ (x_1,\, x_2) \in E_1 \times E_2}} d_Y(f(x_1),\, f(x_2)) = 0. \tag{3.7}$$

When we use the Lemma, we will use a contrapositive version because assuming the discontinuity of α at $\bar{x}$ is not removable amounts to assuming that $\lim\limits_{x \to \bar{x},\, x \in E} f(x)$ does *not* exist, where $X = [a,\, b]$, $E_1 = [a,\, \bar{x})$ and $E_2 = (\bar{x},\, b]$. To *prove* the Lemma we will have to introduce a metric on $E_1 \times E_2$.

To complete the proof of Case (II) we use (3.6), with α instead of f, $Y = \mathbb{R}$, $d_Y(y_1,\, y_2) = |y_1 - y_2|$, $X = [a,\, b]$, $d_X(x_1,\, x_2) = |x_1 - x_2|$ and $E_1 = [a,\, \bar{x})$, $E_2 = (\bar{x},\, b]$. Since $\lim\limits_{x \to \bar{x},\, x \in E} f(x)$ does not exist there exists $\zeta > 0$ such that for all $\delta > 0$ there exist $t' < \bar{x} < t''$, with $t' \in [a,\, \bar{x})$ and $t'' \in (\bar{x},\, b]$, with $t'' - t' < \delta$ and such that $|\alpha(t'') - \alpha(t')| \geq \zeta$. As a technical matter, we apply the lemma starting with $\delta/2$ instead of δ. Then $t'' - t' = t'' - \bar{x} + \bar{x} - t' < \delta/2 + \delta/2 = \delta$. For some of you, “we apply the lemma” is something you may need to check carefully, or ask about! At this point we can return to “(we must verify our assertion)” at the top of the page and see that the proof is indeed complete (again, subject to some careful checking on your part!).

Proof of Lemma (3.6)

Before we actually begin the proof let us write $p = (p_1,\, p_2)$ (and so on) and define $d_2(p,\, q) := d_X(p_1,\, q_1) + d_X(p_2,\, q_2)$ when p and q belong to $E_1 \times E_2$. We show next that d_2 is a metric on $E_1 \times E_2$. It is immediate from the definition that $0 \leq d_2(p,\, q) < \infty$ and that $d_2(q,\, p) = d_2(p,\, q)$ when p and q belong to $E_1 \times E_2$. If $d_2(p,\, q) = 0$ then $d_X(p_1,\, q_1) + d_X(p_2,\, q_2) = 0$ so $d_X(p_1,\, q_1) = 0 = d_X(p_2,\, q_2)$ and thus $p_1 = q_1$ and $p_2 = q_2$ so $p = q$. We have $d_2(p,\, r) + d_2(r,\, q) = d_X(p_1,\, r_1) + d_X(p_2,\, r_2) + d_X(r_1,\, q_1) + d_X(r_2,\, q_2) \geq d_X(p_1,\, q_1) + d_X(p_2,\, q_2) = d_2(p,\, q)$, so the triangle inequality holds.

We can begin the proof of (3.6) now. If $y := \lim\limits_{x \to \bar{x},\, x \in E} f(x)$ exists then for all $\epsilon > 0$ there exists $\delta > 0$ such that $d_X(x,\, \bar{x}) < \delta/2 \Rightarrow d_Y(f(x),\, y) < \epsilon/2$. Hence $d_Y(f(p_1),\, f(p_2)) < \epsilon$ if $d_2(p,\, (\bar{x},\, \bar{x})) < \delta/2$. Thus (3.7) holds.

Next we suppose that (3.7) holds. We will use the completeness of Y and the “Sequences are good enough” Theorem **(4.2)** in Rudin’s book. Let us use the notation $\bar{\bar{x}}$ for $(\bar{x},\, \bar{x})$. We are given that for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < d_2(p,\, \bar{\bar{x}}) < \delta \Rightarrow d_Y(f(p_1),\, f(p_2)) < \epsilon/2$. If $0 < d_X(p_1,\, \bar{x}) < \delta/2$, $0 < d_X(p_1',\, \bar{x}) < \delta/2$

and $0 < d_X(p_2, \bar{x}) < \delta/2$ (all of these inequalites can hold since $\bar{x}$ is a limit point of E_1 and of E_2) then $d_Y(f(p_1), f(p_1')) \le d_Y(f(p_1), f(p_2)) + d_Y(f(p_2), f(p_1')) = d_Y(f(p_1), f(p_2)) + d_Y(f(p_1'), f(p_2)) < \epsilon$. We have here used the points (p_1, p_2) and (p_1', p_2) in $E_1 \times E_2$. We have shown: $0 < d_X(p_1, \bar{x}) < \delta/2$ and $0 < d_X(p_1', \bar{x}) < \delta/2$ implies $d_Y(f(p_1), f(p_1')) < \epsilon$, where $\delta > 0$ is obtained by applying (3.7), with $\epsilon > 0$ given.

Now suppose $\{p_{1k}\}$ is a sequence in E_1 that converges to $\bar{x}$. There exists K so large that $k \ge K \Rightarrow d_X(p_{1k}, \bar{x}) < \delta/2$. Thus if also $k' \ge K$, $d_Y(f(p_{1k}), f(p_{1k'})) < \epsilon$. Hence $\{f(p_{1k})\}$ is Cauchy in Y and so converges to $\bar{y} \in Y$. A sequence $\{p_{2k}\}$ in E_2 converging to $\bar{x}$ is needed too. We should use a specific sequence of ϵ's, say $1/2^n$.

To show that $\bar{y}$ does not depend on the sequence $\{p_{1k}\}$ we consider another sequence $\{p_{1k}'\}$ in E_1 that converges to $\bar{x}$. Then, as we saw before, $d_Y(f(p_{1k}), f(p_{1k}')) \le d_Y(f(p_{1k}), f(p_{2k})) + d_Y(f(p_{2k}), f(p_{1k}')) \to 0$ as $k \to \infty$.

By **(4.2)** $\lim\limits_{x \to \bar{x},\, x \in E} f(x)\ (= \bar{y})$ exists. The proof is done.

An Example

As promised in Section 2, we will construct functions α and f on $[-1, 1]$ such that $\int_{-1}^{0} f\,d\alpha$ and $\int_{0}^{1} f\,d\alpha$ both exist but $\int_{-1}^{1} f\,d\alpha$ does not exist, because of a common discontinuity at 0. We take $f(x) := 0$ if $x < 0$ and $f(x) := 1/2$ if $x \ge 0$. We define $\alpha(x) := 1 + x$ on $[-1, 0]$ and define $\alpha(x) := x$ on $(0, 1]$. By Theorem (3.2) $\int_{-1}^{1} f\,d\alpha$ does not exist because f and α are both discontinuous at 0. To see that $\int_{-1}^{0} f\,d\alpha$ exists we guess that the value of $\int_{-1}^{0} f\,d\alpha$ is zero. To check this we let $\epsilon > 0$ be given and consider π with $n_\pi = n$ and $\mathrm{mesh}\,(\pi) < 2\epsilon$.

Then $$|RS(f, \alpha, [-1, 0], \pi, \xi) - 0| = f(\xi_n)(1 - (1 + x_{n-1})) = |f(\xi_n)|(-x_{n-1}) \le \frac{1}{2}\mathrm{mesh}\,(\pi) < \epsilon.$$

The only possible nonzero value that f can have is $1/2$ and that occurs when (and only when) $\xi_n = 0$. Thus $\int_{-1}^{0} f\,d\alpha$ exists and has value zero.

To see that $\int_{0}^{1} f\,d\alpha$ exists we don't guess. Instead we examine a Riemann-Stieltjes sum. We let $\epsilon > 0$ be given and consider π with $n_\pi = n > 1$. Then

$$RS(f, \alpha, [0, 1], \pi, \xi) = \frac{1}{2}(\alpha(x_1) - \alpha(0)) + \frac{1}{2}\sum_{i=2}^{n} \Delta x_i = \frac{1}{2}(x_1 - 1) + \frac{1}{2}(1 - x_1) = 0.$$

Thus $\int_{0}^{1} f\,d\alpha$ exists and has value 0.

The second theorem

Let us look at two special examples. First suppose that $\alpha(x) \equiv 1$. Then $\int_{a}^{b} f\,d\alpha$ exists for all functions f defined on $[a, b]$ that take values in a vector space and $\int_{a}^{b} f\,d\alpha = 0$ since every Riemann-Stieltjes sum is zero. Next let $\alpha(x) \equiv 1$ if $x \ne 0$ and let $\alpha(0) = 0$. Then if $a < 0 < b$ and f is continuous at 0, it turns out that $\int_{a}^{b} f\,d\alpha$ exists and $\int_{a}^{b} f\,d\alpha = 0$. For other values of x in $[a, b]$ $f(x)$ must be finite-valued, but f does not have to be bounded in $[a, b]$. The details are left to you.

The limitations placed on the functions f thus have something to do with points at which the functions change their values.

(3.8) **Definition:** *If $\alpha(x)$ is defined on $[a, b]$, we denote by $\Omega = \Omega(\alpha, [a, b])$ the set of all $c \in [a, b]$ such that for all $\delta > 0$ there exists $x \in [a, b]$ with $|x - c| < \delta$ and $\alpha(x) - \alpha(c) \ne 0$.*

(3.9) **Exercise:** *Prove that $\Omega(\alpha, [a, b])$ is closed.*

The second necessary condition for the existence of $\int_{a}^{b} f\,d\alpha$ has to do with the boundedness of f and of α.

(3.10) **Theorem:** *If f is Riemann-Stieltjes integrable on $[a, b]$ with respect to α then f is bounded on $\Omega(\alpha, [a, b])$ and α is bounded on $\Omega(f, [a, b])$*

Proof: We will show first that f is bounded on $\Omega(\alpha, [a, b])$. Suppose not. Then there exists a sequence $\{t_n\}$ in $\Omega := \Omega(\alpha, [a, b])$ such that $|f(t_n)| > n$. Since $f(x)$ is finite at every point x in Ω, there are infinitely many distinct t_n, so some subsequence (that we will still denote $\{t_n\}$) converges to a point $\bar{x}$ in Ω. We may assume that no $t_n = \bar{x}$.

We now choose $\epsilon = 1$ in the definition of Riemann-Stieltjes integrability, and obtain a corresponding $\delta > 0$. For convenience we replace δ by another, possibly smaller, $\delta < (b-a)/2$. We can then construct a partition π_o with mesh size less than δ in such a way that $\bar{x}$ is the left endpoint of some interval I_{i_o} of π_o. We note that $n_{\pi_o} > 2$ and we notice that $\bar{x}$ is also an endpoint of I_{i_o-1} or of I_{i_o+1}.

We know that every neighborhood of $\bar{x}$ contains infinitely many of the t_n. Thus at least one of I_{i_o-1}, I_{i_o} and I_{i_o+1} contains infinitely many of the points t_n. Without loss we assume that I_{i_o} contains infinitely many t_n.

Since Ω is closed we know that I_{i_o} contains a point $\hat{x}$ with $|\alpha(\hat{x}) - \alpha(\bar{x})| > 0$. We know (Why?) that $\hat{x} > \bar{x}$. We add $\hat{x}$ to π_o, giving us a new partition π with $\text{mesh}(\pi) < \delta$. We will now call $[\bar{x}, \hat{x}]$, which is an interval of π, $\hat{I}$. Next we pick the components ξ_i of a selection vector ξ for π in an arbitrary way when $I_i \neq \hat{I}$. We let $\hat{\xi}$ denote the component of ξ in $\hat{I}$, and we let it (i.e., $\hat{\xi}$) be some $x_N \in \hat{I}$. Then $|R(f, \pi, \xi) - RSI| < 1$. We next modify ξ by changing *only* $\hat{\xi} = x_N$ to $\hat{\xi}' := x_M$, where $x_M \in \hat{I}$ and $M >> N$. We call the new selection vector ξ'. Then $|R(f, \pi, \xi') - RSI| < 1$ and

$$RS(f, \pi, \xi') - RSI = RS(f, \pi, \xi) - RSI + (f(x_M) - f(x_N))(\alpha(\hat{x}) - \alpha(\bar{x})).$$

By choosing M very large compared to N we can arrange that $|f(x_M) - f(x_N)||\alpha(\hat{x}) - \alpha(\bar{x})| > 2$.

If $(f(x_M) - f(x_N))(\alpha(\hat{x}) - \alpha(\bar{x})) > 0$, then

$$RS(f, \pi, \xi') - RSI > RS(f, \pi, \xi) - RSI + 2 > -1 + 2 = 1.$$

We argue similarly if $(f(x_M) - f(x_N))(\alpha(\hat{x}) - \alpha(\bar{x})) < 0$. The definition of Riemann-Stieltjes integrability is contradicted. Hence f is bounded on $\Omega(\alpha, [a, b])$ if $\int_a^b f\, d\alpha$ exists.

Since the existence of $\int_a^b f\, d\alpha$ implies that of $\int_a^b \alpha\, df$, we can reverse the rôles of f and α, so the rest of the Theorem follows.

Remark

When $\alpha(x) = x$, $\Omega = [a, b]$ for every compact interval $[a, b]$. Thus if f is Riemann integrable on $[a, b]$, f is bounded on $[a, b]$.