§4 Functions of bounded variation: definition and properties

In what we do from now on, at least one of f and α will be a function of bounded variation, unless otherwise stated. We will begin by discussing real-valued functions of bounded variation. This material can also be found in Chapter 2 of *Measure and Integral,* by Wheeden and Zygmund. Our immediate goal is to show that a function, such as α, has bounded variation if and only if it can be expressed as the difference between two increasing functions. This can connect us back to Rudin's book.

(4.1) **Definition:** *A function* $f : [a, b] \to \mathbb{R}$ *is a function of bounded variation on* $[a, b]$ *if*

$$V = V(f, [a, b]) := \sup_{\pi|[a,b]} \sum_{1}^{n_\pi} |f(x_i) - f(x_{i-1})| < \infty \text{ and we say that } f \in BV[a, b].$$

We call V the *variation,* or *total variation,* of f over $[a, b]$. We call $BV[a, b]$ the space of functions of bounded variation on $[a, b]$. To go farther it will be useful to have some more notation. If π is a partition of $[a, b]$ we will write

$$(4.2) \quad V_\pi = V_\pi(f, [a, b]) := \sum_{1}^{n_\pi} |f(x_i) - f(x_{i-1})|, \text{ so that } V = \sup_{\pi|[a,b]} V_\pi \text{ (here, } f \text{ and } [a, b] \text{ are "assumed").}$$

We can call V_π the "π-variation" of f over $[a, b]$. Since f has a finite value for each $\pi|[a, b]$, V_π is always finite. However, V can be infinite. This is so, for example, if f is the Dirichlet function.

Each V_π pays attention *only* to the absolute value of the *difference* between the values at the opposite ends of an interval of the partition π. We will need to take the *signs* of those differences into account, and they will lead to two new "variations."

For a real number x we define its *positive part* to be $x^+ := \max\{0, x\}$ and we define its *negative part* to be $x^- := \max\{0, -x\}$. Both "parts" are non-negative, and we have $x^+ + x^- = |x|$ and $x^+ - x^- = x$.

(4.3) **Exercise:** *Prove that for all real numbers* x *and* y, $(x + y)^+ \le x^+ + y^+$ *and* $(x + y)^- \le x^- + y^-$. *These are "triangle inequalities!" What can be said about* $(xy)^+$ *and* $(xy)^-$? x^+x^-?

We now define the "positive" and "negative" "π-variations" of f over $[a, b]$:

$$(4.4) \qquad P_\pi = P_\pi(f, [a, b]) := \sum_{1}^{n_\pi} (f(x_i) - f(x_{i-1}))^+ \text{ and } N_\pi = N_\pi(f, [a, b]) := \sum_{1}^{n_\pi} (f(x_i) - f(x_{i-1}))^-.$$

(4.5) **Definition:** *The positive variation,* $P = P(f, [a, b])$ *and the negative variation* $N = N(f, [a, b])$ *of* f *over* $[a, b]$ *are given by* $P = \sup_{\pi|[a,b]} P_\pi$ *and* $N = \sup_{\pi|[a,b]} N_\pi$ *respectively.*

For example, if f increases on $[a, b]$, $P_\pi = V_\pi = f(b) - f(a)$ and $N_\pi = 0$. If we look at $f(x) := |x|$ on $[-1, 1]$ we will always have $0 \le P_\pi \le 1$ and $0 \le N_\pi \le 1$, and $0 \le V_\pi \le 2$.

Because of how x^+ and x^- were defined, we always have (for *any* function)

$$(4.6) \qquad P_\pi + N_\pi = V_\pi \text{ and } P_\pi - N_\pi = f(b) - f(a) \text{ (telescoping sums!).}$$

If τ is a refinement of π, we always have $O_\pi \le O_\tau$, where O stands for any of the letters N, P or V. This follows from several applications of the triangle inequality.

Some properties of functions of bounded variation

(4.7) *If* $f \in BV[a, b]$ *then* f *is bounded on* $[a, b]$.
Proof: Suppose $a \le x \le b$. Then, if we let $\pi := \{a, x, b\}$,

$$|f(x)| = |f(x) - f(a) + f(a)| \le |f(a)| + |f(x) - f(a)| + |f(b) - f(x)| = |f(a)| + V_\pi \le |f(a)| + V.$$

(4.8) *The space* $BV[a, b]$ *is a vector space.*
For all $c \in \mathbb{R}$ *and all* $f \in BV[a, b]$, $V(cf, [a, b]) = |c|V(f, [a, b])$.
For all $f \in BV[a, b]$ *and* $g \in BV[a, b]$, $V(f + g, [a, b]) \le V(f, [a, b]) + V(g, [a, b])$; $V(f, [a, b]) = 0$ *if and only if* f *is constant.*
Proof: The second assertion follows from these facts: for all $\pi|\,[a, b]$, $V_\pi(cf, [a, b]) = |c|V_\pi(f, [a, b])$; $\sup\{|c|x : x \in E\} = |c|\sup\{x : x \in E\} = |c|\sup E$. The first assertion and the first part of the third one follow from the second one and the triangle inequality. Finally, suppose that $V(f, [a, b]) = 0$ and that $a \le x \le b$. Then, with $\pi := \{a, x, b\}$, $|f(x) - f(a)| \le |f(x) - f(a)| + |f(b) - f(x)| = V_\pi \le V = 0$. Therefore $f(x) \equiv f(a)$.

(4.9) *If* $f \in BV[a, b]$ *and* $a < c < b$ *then* $f \in BV[a, c]$ *and* $f \in BV[c, b]$, *and conversely.*
Moreover, $V = V(f, [a, b]) = V(f, [a, c]) + V(f, [c, b])$.

Proof: If $f \in BV[a, b]$ and $a < c < b$, let partitions $\sigma|\,[a, c]$ and $\tau|\,[c, b]$ be given. Then $\pi := \sigma \cup \tau$ is a partition of $[a, b]$ so $V_\sigma + V_\tau = V_\pi \le V$, hence $V_\sigma \le V$ and $V_\tau \le V$. Thus $f \in BV[a, c]$ and $f \in BV[c, b]$. Conversely, suppose that $a < c < b$ and that $f \in BV[a, c]$ and $f \in BV[c, b]$. Let $\pi|\,[a, b]$. Then $\pi_c := \pi \cup \{c\}$ is a refinement of π. Therefore $V_\pi \le V_{\pi_c} = V_\sigma + V_\tau$, where $\sigma := \pi_c \cap [a, c]$ and τ is defined similarly. By hypothesis, $V_\pi \le V_{\pi_c} = V_\sigma + V_\tau \le V(f, [a, c]) + V(f, [c, b])$. Thus $V(f, [a, b]) \le V(f, [a, c]) + V(f, [c, b]) < \infty$. This proves part of the asserted equality. To show the other inequality, now that we know $V < \infty$ let partitions $\sigma|\,[a, c]$ and $\tau|\,[c, b]$ be given. We recall that earlier we had $V_\sigma + V_\tau = V_{\pi_c} \le V$, so $V_\sigma + V_\tau \le V$ whenever $\sigma|\,[a, c]$ and $\tau|\,[c, b]$ were arbitrary partitions of $[a, c]$ and $[c, b]$, respectively.

Thus $\sup\limits_{\sigma|\,[a, c]} (V_\sigma + V_\tau) = V(f, [a, c]) + V_\tau \le V$, and so $\sup\limits_{\tau|\,[c, b]} (V(f, [a, c]) + V_\tau) = V(f, [a, c]) + V(f, [c, b]) \le V$.

Please note that the first inequality holds for an *arbitrary* $\tau|\,[c, b]$, making the second one valid.

(4.10) **Exercise:** *Prove that the equalities in* (4.6) *and the statement following it hold for every function* $f : [a, b] \to \mathbb{R}$, *whether* f *is a function of bounded variation or not.*

Motivated by (4.5) and (4.6), when $f : [a, b] \to \mathbb{R}$ and $a \le x \le b$ we can define the three functions

$$V(x) := V(f, [a, x]), \quad P(x) := P(f, [a, x]) \text{ and } N(x) := N(f, [a, x]). \tag{4.11}$$

We do this by taking suprema over partitions τ of $[a, x]$ of $V_\tau(f, [a, x])$, $P_\tau(f, [a, x])$ and $N_\tau(f, [a, x])$. Each of these is an increasing function of x. Jordan's Theorem asserts that if $f \in BV[a, b]$ we can represent f in terms of $P(x)$ and $N(x)$.

(4.12) **Theorem (Jordan):** *A function* $f \in BV[a, b]$ *if and only if there exist functions* g *and* h, *both increasing on* $[a, b]$, *such that* $f(x) = g(x) - h(x)$ *for* $a \le x \le b$. *If this is the case, then* $P(x) \le g(x) - g(a)$, $N(x) \le h(x) - h(a)$ *and* $f(x) = f(a) + P(x) - N(x)$ *for* $a \le x \le b$.

Proof: Suppose first that $f(t) = g(t) - h(t)$, $t \in [a, b]$, where the functions g and h are both increasing on $[a, b]$. Let $\pi\,|[a, b]$ (later, we will apply this when $x \in [a, b]$ and $\pi\,|[a, x]$). Then

$$\Delta f_i = f(x_i) - f(x_{i-1}) = \Delta g_i - \Delta h_i \begin{cases} \le & \Delta g_i \\ \ge & -\Delta h_i. \end{cases} \tag{4.13}$$

Thus $-\Delta h_i \le \Delta f_i \le \Delta g_i$, so $|\Delta f_i| \le \max\{\Delta g_i, \Delta h_i\} \le \Delta g_i + \Delta h_i$ for $1 \le i \le n_\pi$. Hence $V_\pi(f) \le V_\pi(g) + V_\pi(h) = g(b) - g(a) + h(b) - h(a) < \infty$, so $f \in BV[a, b]$.

Next, we show that $f(x) = f(a) + P(x) - N(x)$ for $a \le x \le b$. But we will do this just by showing it for $x = b$. Then we can use (4.11) and let each $x \in [a, b]$ play the rôle of b. This will show the existence of the functions $g(x)(= f(a) + P(x))$ and $h(x)(= N(x))$. We will use (4.6) and (4.7). After that is done, we'll prove the P–g and N–h inequalities.

By the definitions of P, N and V we know there exist sequences $\{\pi_k\}$, $\{\rho_k\}$ and $\{\sigma_k\}$ such that $P_{\pi_k} \to P$, $N_{\rho_k} \to N$ and $V_{\sigma_k} \to V$. Let us define $\tau_k := \pi_k \cup \rho_k \cup \sigma_k$. By (4.7) $P_{\pi_k} \le P_{\tau_k} \le P$. By the Squeeze Principle $P_{\tau_k} \to P$. Similarly, $N_{\tau_k} \to N$ and $V_{\tau_k} \to V$. By (4.6) and Limit Theorems

$$P + N = V \text{ and } P - N = f(b) - f(a) \text{ and the second is the same as } f(x) = f(a) + P(x) - N(x)$$

when $x = b$. By (4.9) we can use any $x \in [a, b]$ in place of b by restricting our attention to f on $[a, x]$.

Now suppose that $f(x)$ is *defined* as the difference of two increasing functions on $[a, b]$: $f(t) = g(t) - h(t)$. We will use (4.13) again, as well as the following observation: $t \mapsto t^+$ is increasing and $t \mapsto t^-$ is decreasing. Therefore, with the help of (4.13), applied to partitions of $[a, x]$, $(\Delta f_i)^+ \leq (\Delta g_i)^+ = \Delta g_i$ and $\Delta h_i = (-\Delta h_i)^- \geq (\Delta f_i)^-$. Hence $P_\pi(f, [a, x]) \leq P_\pi(g, [a, x]) = g(x) - g(a)$. Similarly, $h(x) - h(a) = N_\pi(-h, [a, x]) \geq N_\pi(f, [a, x])$. When, in each case, we take the supremum over all $\pi \,|\, [a, x]$, we get $P(x) \leq g(x) - g(a)$ and $N(x) \leq h(x) - h(a)$. These inequalities "say" that there is no "wasted cancellation" in the formula $f(x) = f(a) + P(x) - N(x)$.