§5 Some Riemann-Stieltjes integrals that exist

Theorems (5.1), (5.8) and (5.10) give *sufficient* conditions on f and on α, on a bounded interval $[a, b]$, that guarantee the existence of $\int_a^b f\,d\alpha$. Two (classes of) examples and some other results are here too.

(5.1) **Theorem:** *If $[a, b]$ is a bounded closed interval, f is continuous on $[a, b]$ and α is a function of bounded variation on $[a, b]$ then $\int_a^b f\,d\alpha$ exists and* $\left|\int_a^b f\,d\alpha\right| \le \max_{x\in[a,\,b]} |f(x)|V(\alpha, [a, b])$.

The estimate follows from the same estimate for Riemann-Stieltjes sums, valid by our hypotheses, and the existence of the integral as a limit, as seen in the proof of the Cauchy Criterion for Riemann-Stieltjes integrability, (1.7). To prove the existence statement, we will also use (1.7). Thus we need a usable way to estimate the difference between two Riemann-Stieltjes sums. This is what the upcoming Lemma gives. We need a useful definition and some notation.

(5.2) **Definition:** *If U is an interval and f is a function defined on U we define the oscillation of f on U, denoted $\omega(f, U)$, by*

$$\omega(f, U) := \sup_{x,\,y\,\in U} |f(x) - f(y)|.$$ *We allow the interval to be open or half-open as well as closed.*

We will let $\omega_i = \omega_i(f) = \omega(f, I_i)$ when I_i is an interval (closed!) of a partition π.

Now we are ready for the Lemma. It is *officially* stated for real-valued functions, but can also be used in the more general contexts. Note that its hypotheses allow the oscillations of f and the variation of α to be infinite.

(5.3) **Lemma:** *If $[a, b]$ is a bounded closed interval, f and α are functions defined on $[a, b]$, π and π' are partitions of $[a, b]$, then*

(5.4) $$|RS(f, \alpha, \pi) - RS(f, \alpha, \pi')| \le \sum_{i=1}^{n_\pi} \omega_i\, V(\alpha, I_i) + \sum_{j=1}^{n_{\pi'}} \omega'_j\, V(\alpha, I'_j) = \sum_{i=1}^{n_\pi} \omega_i\, \Delta V_i + \sum_{j=1}^{n_{\pi'}} \omega'_j\, \Delta' V_j,$$

where ω_i and ω'_j denote the oscillations of f on the intervals I_i and I'_j of the partitions π and π', respectively.

In the last part of (5.4), with $V(x) := V(\alpha, [a, x])$, we defined $\Delta V_i = V(\alpha, I_i) = V(x_i) - V(x_{i-1})$ and we defined $\Delta' V_j = V(\alpha, I'_j) = V(x'_j) - V(x'_{j-1})$, but that part really makes sense only when $V(b) < +\infty$.

First we will use Lemma (5.3) to prove Theorem (5.1) and then prove Lemma (5.3).

According to our basic Cauchy Criterion for Riemann-Stieltjes Integrability, a function f on an interval $[a, b]$ is Riemann-Stieltjes integrable if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all partitions π, π',

(5.5) $$\text{if } \operatorname{mesh}(\pi) < \delta \text{ and } \operatorname{mesh}(\pi') < \delta, \text{ then } |RS(f, \alpha, \pi) - RS(f, \alpha, \pi')| < \epsilon.$$

Let $\epsilon > 0$ be given. Let V denote $V(\alpha, [a, b]) < \infty$. Since f is continuous on $[a, b]$ f is *uniformly* continuous on $[a, b]$. Hence there exists $\delta > 0$ such that for all $u, v \in [a, b]$, $|u - v| < \delta \Rightarrow |f(u) - f(v)| < \epsilon/(2V + 1)$. This implies that each $\omega_i \le \epsilon/(2V + 1)$. Now we can see (by (5.4)!) that if the *condition* in (5.5) is true,

$$|RS(f, \alpha, \pi) - RS(f, \alpha, \pi')| \le \sum_{i=1}^{n_\pi} \omega_i \Delta V_i + \sum_{j=1}^{n_{\pi'}} \omega'_j \Delta' V_j \le \frac{\epsilon}{2V + 1} \left(\sum_{j=1}^{n_\pi} V(\alpha, I_i) + \sum_{j=1}^{n_{\pi'}} V(\alpha, I'_j) \right) < \epsilon.$$

It may take you a moment, looking back and forth between (5.5) and (5.4) to agree. The proof of (5.1) is done.

Proof of **Lemma** (5.3): We will use the partition τ that consists of all the points of π and of π', the *common refinement* of π and π'. We can write $\tau = \pi \cup \pi'$ to express this. We will denote the points of τ by t_k and the intervals of τ by K_k. Then for each i we can write

(5.6) $$\Delta\alpha_i := \alpha(x_i) - \alpha(x_{i-1}) = \sum_{K_k \subseteq I_i} \Delta\alpha_k \quad \text{(tricky notation! Here } \Delta\alpha_k := \alpha(t_k) - \alpha(t_{k-1}) \text{ refers to } K_k).$$

That is, the difference $\Delta\alpha_i$ is the sum of the differences $\Delta\alpha_k$, on the intervals K_k of τ that are contained in I_i. The formula (5.6) is true because each interval K_k of τ is contained in exactly one interval $I_{i(k)}$ of π. In other words, $i(k)$ is the i such that $K_k \subseteq I_{i(k)}$. There is a similar formula for $\Delta\alpha_j$ and a similar function $j(k)$.

Now we need to bring in arbitrary "selection vectors" ξ and ξ' to construct our Riemann sums. Thus for $RS(f,\pi)$ we write

$$RS(f,\pi,\xi) = \sum_{i=1}^{n_\pi} f(\xi_i)\Delta\alpha_i = \sum_{i=1}^{n_\pi} f(\xi_i) \sum_{K_k\subseteq I_i} \Delta\alpha_k = \sum_{k=1}^{n_\tau} f(\xi_{i(k)})\Delta\alpha_k; \quad \text{the last sum is } \textit{not} \text{ an } RS(f,\tau,\xi'')!$$

The last sum may need some explanation! Each interval K_k is contained in one and only one of the intervals I_i. This i we called $i(k)$. We can't reverse this though, because each i can be associated with several k. Finally, each k gets used, exactly once. This allows us to express $RS(f,\,\pi,\,\xi)$ in a way that will become compatible with $RS(f,\,\pi',\,\xi')$. Again, $i(k)$ is the i such that $K_k \subseteq I_{i(k)}$. Notice that $\xi_{i(k)} \in I_{i(k)}$ need not belong to $K_k \subseteq I_{i(k)}$!

Now it is time to be clever. We choose a selection vector ξ'' for τ. Here, $\xi''_k \in K_k$ for every k. Then

$$RS(f,\pi,\xi) - RS(f,\tau,\xi'') = \sum_{k=1}^{n_\tau} \big(f(\xi_{i(k)}) - f(\xi''_k)\big)\Delta\alpha_k.$$

We can regroup this sum once again, to return to the point of view of π:

$$RS(f,\pi,\xi) - RS(f,\tau,\xi'') = \sum_{i=1}^{n_\pi} \sum_{K_k\subseteq I_i} \big(f(\xi_i) - f(\xi''_k)\big)\Delta\alpha_k.$$

Each difference $f(\xi_i) - f(\xi''_k)$ uses two function values whose arguments are both in I_i, so $|f(\xi_i) - f(\xi''_k)| \le \omega_i$. By much use of the triangle inequality our sum can now be replaced by an estimate:

$$|RS(f,\pi,\xi) - RS(f,\tau,\xi'')| \le \sum_{i=1}^{n_\pi} \sum_{K_k\subseteq I_i} \omega_i|\Delta\alpha_k| = \sum_{i=1}^{n_\pi} \omega_i \sum_{K_k\subseteq I_i} |\Delta\alpha_k| \le \sum_{i=1}^{n_\pi} \omega_i V(\alpha,\, I_i). \tag{5.7}$$

We complete the proof of the Lemma using a similar argument involving π' and τ, together with the triangle inequality (the tricky notation in (5.6) is used again):

$$\begin{aligned} |RS(f,\pi,\xi) - RS(f,\pi',\xi')| &\le |RS(f,\pi,\xi) - RS(f,\tau,\xi'')| + |RS(f,\tau,\xi'') - RS(f,\pi',\xi')| \\ &\le \sum_{i=1}^{n_\pi} \omega_i V(\alpha,\, I_i) + \sum_{j=1}^{n_{\pi'}} \omega'_j V(\alpha,\, I'_j). \end{aligned}$$

(5.8) **Theorem:** *If $[a,\,b]$ is a bounded closed interval, f is continuous on $[a,\,b]$ and α' is continuous on $[a,\,b]$ then $\int_a^b f\,d\alpha$ exists and* $\displaystyle\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$. I.e., $d\alpha(x) = \alpha'(x)\,dx$.

Proof: By Theorem (5.1) the integral on the right exists because the *integrand,* $f(x)\alpha'(x)$, is continuous and $\alpha(x) = x$ is in $BV[a,\,b]$. We consider a Riemann-Stieltjes sum (we'll use the mean Value Theorem)

$$\begin{aligned} RS(f,\pi,\xi) &= \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^{n_\pi} f(\xi_i)\alpha'(\eta_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n_\pi} f(\xi_i)\alpha'(\xi_i)\Delta x_i + \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha'(\eta_i) - \alpha'(\xi_i))\Delta x_i \qquad (5.9) \\ &= R(f\alpha',\,\pi,\,\xi) + \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha'(\eta_i) - \alpha'(\xi_i))\Delta x_i, \end{aligned}$$

where the first sum is a Riemann sum and the second contains terms that have *two* small factors. This leads to the desired result. The details are as follows.

We have already noticed that $R(f\alpha', \pi, \xi) \to \int_a^b f(x)\alpha'(x)\,dx$ as $\mathrm{mesh}(\pi) \to 0$. Thus given $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{mesh}(\pi) < \delta$ implies $|R(f\alpha', \pi, \xi) - \int_a^b f(x)\alpha'(x)\,dx| < \epsilon/2$. Since α' is uniformly continuous there exists $\delta' > 0$ such that for all $x, y \in [a, b]$, $|x-y| < \delta'$ implies that $|\alpha(x)-\alpha(y)| < \epsilon/[2(b-a)(1+\max_{x\in[a,b]}|f(x)|)]$. If $\mathrm{mesh}(\pi) < \min\{\delta, \delta'\}$ then $|\alpha'(\eta_i) - \alpha'(\xi_i)| < \epsilon/[2(b-a)(1+\max_{x\in[a,b]}|f(x)|)]$ and so (in (5.9))

$$RS(f,\pi,\xi) - \int_a^b f(x)\alpha'(x)\,dx = \left(R(f\alpha', \pi, \xi) - \int_a^b f(x)\alpha'(x)\,dx\right) + \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha'(\eta_i) - \alpha'(\xi_i))\Delta x_i.$$

Each of the two main terms on the right has absolute value less than $\epsilon/2$. Thus by the "original" definition of Riemann-Stieltjes integrability, the proof is complete.

Some further results and examples

We recall the notation $\omega_i := \omega_i(f) = \omega(f, I_i)$ from (5.2).

(5.10) **Theorem (An Oscillation Criterion):** *Suppose that* $\alpha \in BV[a, b]$. *If* f *is defined on* $[a, b]$ *and*

(5.11) *for all positive* β *and* η *there exists* $\delta > 0$ *such that* $$\mathrm{mesh}(\pi) < \delta \Rightarrow \sum_{\omega_i > \eta} V(\alpha, I_i) < \beta,$$

then f *is Riemann-Stieltjes integrable on* $[a, b]$.

Proof: By Lemma (5.3), for any two Riemann-Stieltjes sums over partitions π and π' of $[a, b]$,

$$|RS(f, \alpha, \pi) - RS(f, \alpha, \pi')| \le \sum_{i=1}^{n_\pi} \omega_i\,\Delta V_i + \sum_{j=1}^{n_{\pi'}} \omega'_j\,\Delta' V_j, \tag{5.12}$$

where we have used the notation ΔV_i and $\Delta' V_j$ to stand for $V(\alpha_i, I_i)$ and $V(\alpha_j, I'_j)$ respectively. In terms of this notation, (5.11) becomes

(5.13) for all positive β and η there exists $\delta > 0$ such that $\mathrm{mesh}(\pi) < \delta$ implies that $\sum_{\omega_i > \eta} \Delta V_i < \beta$.

Let $\epsilon > 0$ be given. In (5.12) let us rewrite the first of the sums on the right:

$$\sum_{i=1}^{n_\pi} \omega_i\,\Delta V_i = \sum_{\omega_i > \eta} \omega_i\,\Delta V_i + \sum_{\omega_i \le \eta} \omega_i\,\Delta V_i \le \sum_{\omega_i > \eta} \omega_i\,\Delta V_i + \eta V, \quad \text{where } V := V(\alpha, [a, b]). \tag{5.14}$$

Admittedly, replacing the second sum by ηV is an overestimate. We will simply choose $\eta := \dfrac{\epsilon}{4(V+1)}$. Next, we choose $\beta := \epsilon/4$, obtain $\delta > 0$ such that (5.13) holds for our η and β. Finally, we assume that $\mathrm{mesh}(\pi) < \delta$ and $\mathrm{mesh}(\pi') < \delta$ and obtain the estimate $< \epsilon/2$ in (5.14) for both sums in (5.12) and conclude that $\int_a^b f\,d\alpha$ exists. Be sure to double-check the details!

(5.15) **Theorem:** *Suppose that* f *and* α *are real-valued functions defined on* $[a, b]$ *and that* $\alpha \in BV[a, b]$. *Let* $V(x) := V(\alpha, [a, x])$ *and suppose that* $\int_a^b f\,dV$ *exists. Then* $\int_a^b f\,d\alpha$ *exists, as well as (by Jordan's Theorem, (4.12))* $\int_a^b f\,dP$ *and* $\int_a^b f\,dN$.

Proof: By the Cauchy Criterion, we know that for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{mesh}(\pi) < \delta$ implies that $|RS(f, V, \pi, \xi) - RS(f, V, \pi, \xi')| < \epsilon/4$. Here, we applied the Cauchy Criterion with two partitions that are the same. Thus for *arbitrary* selection vectors ξ and ξ' compatible with π, if $\mathrm{mesh}(\pi) < \delta$ then

$$\epsilon/4 > |RS(f, V, \pi, \xi) - RS(f, V, \pi, \xi')| = \left|\sum_{i=1}^{n_\pi}(f(\xi_i) - f(\xi'_i))\Delta V_i\right|. \tag{5.16}$$

We will now very carefully choose the selection vectors. To do this we work with one interval I_i at a time. We choose ξ_i and ξ_i' in I_i so that $f(\xi_i) - f(\xi_i') \geq 2\omega_i/3$. This is possible by the definition of ω_i. The product $(f(\xi_i) - f(\xi_i'))\Delta V_i$ is non-negative, so (5.16) becomes (for *all* $\pi|[a, b]$ with $\operatorname{mesh}(\pi) < \delta$)

$$\epsilon/4 > \left|\sum_{i=1}^{n_\pi}(f(\xi_i) - f(\xi_i'))\Delta V_i\right| = \sum_{i=1}^{n_\pi}(f(\xi_i) - f(\xi_i'))\Delta V_i \geq \frac{2}{3}\sum_{i=1}^{n_\pi}\omega_i\Delta V_i. \text{ Thus } \epsilon/2 > \sum_{i=1}^{n_\pi}\omega_i\Delta V_i \text{ if } \operatorname{mesh}(\pi) < \delta.$$

Once again we apply (5.3), now using α instead of V. We now have, if $\operatorname{mesh}(\pi) < \delta$ and $\operatorname{mesh}(\pi') < \delta$, then

$$|RS(f,\, \alpha,\, \pi) - RS(f,\, \alpha,\, \pi')| \leq \sum_{i=1}^{n_\pi}\omega_i\,\Delta V_i + \sum_{j=1}^{n_{\pi'}}\omega_j'\,\Delta' V_j < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus the Cauchy Criterion for the existence of $\int_a^b f\,d\alpha$ is satisfied.

(5.17) **Theorem:** *Suppose that* $\alpha \in BV[a, b]$ *and, for each* $n \in \mathbb{N}$, $\int_a^b f_n\,d\alpha$ *exists. If* $f_n \to f$ *uniformly as* $n \to \infty$ *then* $\int_a^b f\,d\alpha$ *exists and* $\int_a^b f_n\,d\alpha \to \int_a^b f\,d\alpha$ *as* $n \to \infty$.

Proof: We use the Cauchy Criterion. Suppose $\epsilon > 0$ is given. We let V denote $V(\alpha, [a, b])$. We choose N so large that $|f_N(x) - f(x)| < \frac{\epsilon}{4(V+1)}$ for all $x \in [a, b]$. Now if ξ is a selection vector compatible with π,

$$\text{(5.18)} \qquad |RS(f,\, \pi,\, \xi) - RS(f_N,\, \pi,\, \xi)| \leq \sum_{i=1}^{n_\pi}|f(\xi_i) - f_N(\xi_i)||\Delta\alpha_i| < \frac{\epsilon}{4(V+1)}\sum_{i=1}^{n_\pi}|\Delta\alpha_i| \leq \frac{\epsilon V}{4(V+1)} < \epsilon/4.$$

We can proceed similarly if ξ' is a selection vector compatible with π'. Thus we have

$$|RS(f,\, \pi) - RS(f,\, \pi')| \leq |RS(f,\, \pi) - RS(f_N,\, \pi)| + |RS(f_N,\, \pi) - RS(f_N,\, \pi')| + |RS(f_N,\, \pi') - RS(f,\, \pi')|.$$

By (5.18) the first term on the right is less that $\epsilon/4$. A version of (5.18) can be applied to the last term as well. Let us carefully note, in passing, that in (5.18) we used the *same* selection vectors in $RS(f,\, \pi,\, \xi)$ and $RS(f_N,\, \pi,\, \xi)$. We did this to take advantage of the uniform convergence. Thus so far we know that

$$\text{(5.19)} \qquad |RS(f,\, \pi) - RS(f,\, \pi')| < \epsilon/2 + |RS(f_N,\, \pi) - RS(f_N,\, \pi')|.$$

Since $\int_a^b f_N\,d\alpha$ exists there exists $\delta > 0$ such that if $\pi|[a, b]$, $\pi'|[a, b]$, $\operatorname{mesh}(\pi) < \delta$ and $\operatorname{mesh}(\pi') < \delta$, then $|RS(f_N,\, \pi) - RS(f_N,\, \pi')| < \epsilon/2$ for all Riemann-Stieltjes sums over π and π'. Here, we use *arbitrary* selection vectors ξ and ξ' compatible with their respective partitions.

Now we can put the estimate we just obtained from the existence of $\int_a^b f_N\,d\alpha$ into (5.19) in order to find that $|RS(f,\, \pi) - RS(f,\, \pi')| < \epsilon$, so the Cauchy Criterion is satisfied and $\int_a^b f\,d\alpha$ exists.

(5.20) **Exercise:** *Use the existence of* $\int_a^b f\,d\alpha$ *now obtained and* (5.18) *to prove that* $\int_a^b f_n\,d\alpha \to \int_a^b f\,d\alpha$ *as* $n \to \infty$.

The several Exercises and problems that follow cover two classes of examples. In both cases, the functions being integrated are (usually) the continuous functions on $[0, 1]$. The integrators will be functions of bounded variation that are discontinuous on denumerable sets. In the first class the Riemann-Stieltjes integrals are series.

(5.21) **Problem:** Prove the following Theorem. Suggestion: adapt techniques from the proof of (5.17).

Theorem: *Let* f *be bounded on* $[a, b]$. *Suppose that for each* $n \in \mathbb{N}$, $\alpha_n \in BV[a, b]$ *and* $\int_a^b f\,d\alpha_n$ *exists. Suppose also that* $V(\alpha - \alpha_n,\, [a, b]) \to 0$ *as* $n \to \infty$. *Then* $\int_a^b f\,d\alpha$ *exists and* $\int_a^b f\,d\alpha_n \to \int_a^b f\,d\alpha$ *as* $n \to \infty$.

(5.22) **Example:** Suppose $\{t_n\}$ is a sequence of distinct points in $(0, 1)$. For instance, $\{t_n\}$ might be an enumeration of the rational numbers in $(0, 1)$. Let $\sum_n a_n$ be an absolutely convergent series of real numbers. We will define increasing functions g and h on $[0, 1]$, put $\alpha := g - h$ and compute $\int_0^1 f\,d\alpha$ when f is continuous on $[0, 1]$.

We define

$$g(x) := \begin{cases} 0, & \text{if } x = 0; \\ \displaystyle\sum_{t_n < x} a_n^+, & \text{if } 0 < x < 1; \\ \displaystyle\sum_n a_n^+, & \text{if } 0 < x < 1 \end{cases} \quad \text{and} \quad h(x) := \begin{cases} 0, & \text{if } x = 0; \\ \displaystyle\sum_{t_n < x} a_n^-, & \text{if } 0 < x < 1; \\ \displaystyle\sum_n a_n^-, & \text{if } 0 < x < 1. \end{cases}$$

We thus have α defined like g or h but with a_n in place of $a_n^\pm$. You should verify (probably with a bit of staring) that g and h are increasing and bounded and hence that $\alpha \in BV[0, 1]$.

(5.23) **Exercise:** *Show that α is discontinuous on the right at each t_n, but continuous on the left and that $\lim_{x\downarrow t_n} \alpha(x) = \alpha(t_n) + h_n$.*

(5.24) **Exercise:** *Let $\beta_k(x) := 0$ if $0 \le x \le t_k$ and let $\beta_k(x) := a_k$ if $t_k < x \le 1$. Show that $\int_0^1 f\,d\beta_k = a_k f(t_k)$ if f is defined on $[0, 1]$ and continuous at t_k, but that $\int_0^1 f\,d\beta_k$ does not exist if f is discontinuous at t_k.*

With all this work done we can use (5.21) and do some more work to show that $\int_0^1 f\,d\alpha = \sum_n a_n f(t_n)$ if f is continuous on $[0, 1]$.

We define $\alpha_n(x) := \sum_{k\le n} a_k\beta_k(x)$. Then $\alpha_n \to \alpha$ pointwise as $n \to \infty$ (Why?). By (4.8) and induction $V(\alpha_{n+\ell} - \alpha_n, [0, 1]) \le \sum_{k=n+1}^{n+\ell} V(\beta_k, [0, 1]) = \sum_{k=n+1}^{n+\ell} |a_k| \le \sum_{k=n+1}^{\infty} |a_k|$. It takes some work to show that $V(\alpha - \alpha_n, [0, 1]) \le \sum_{k=n+1}^{\infty} |a_k|$, and this last inequality shows that the hypotheses of (5.21) hold. This and (5.24) give us the desired result.

A questioner in class asked how the discontinuities of α were taken care of. In this example, the discontinuities are jumps whose absolute values form a convergent series. Thus in any Riemann-Stieltjes sum, especially in a difference of them, the "misbehavior" due to discontinuities in an interval of a partition is "controlled" by the smallness of series tails and by the continuity of f. The big jumps in α correspond to big values of a_n and there are not many of those.

(5.25) **Problem:** With the same points and the same series used in (5.22) let us define $\alpha(x)$ differently: we let $\alpha(t_n) := a_n$ and we put $\alpha(x) = 0$ otherwise. Show that now $V(\alpha, [0, 1]) = 2\sum_n |a_n|$. Use the ideas behind (5.24) to calculate the Riemann-Stieltjes integral of the related integrators defined using finitely many of the t_n and a_n. Find $\int_0^1 f\,d\alpha$ now, when $f \in C[0, 1]$.