We will investigate Riemann-Stieltjes integrals that are based on "integrators" of bounded variation, so we will begin with a discussion of this kind of function.

On the interval $[0,\, 1]$ the graph of the function $f(x) = |1-2x|$ drops from 1 down to zero then bounces back up to one. As we will see, its "total variation" is 2. Variation measures the "up-and-down" distance traced out by the "point" $f(x)$ as x moves through (in this case) the interval $[0,\, 1]$. If the function has a jump discontinuity we think of the "point" as traveling the distance across the jump "instantly." Thus the Dirichlet function has infinite variation over any subinterval of $[0,\, 1]$ that has positive length.

Definition: *A function $f(x)$ defined on an interval $[a,\, b]$ is a function of bounded variation on $[a,\, b]$ if there exists a number V such that for every partition π of $[a,\, b]$,*

$$\sum_{i=1}^{n_\pi} |f(x_i) - f(x_{i-1})| \le V.$$

We then define the variation of f on $[a,\, b]$ to be the number

$$V_f[a,\, b] := \sup_{\pi \big| [a,\, b]} \sum_{i=1}^{n_\pi} |f(x_i) - f(x_{i-1})|, \tag{1}$$

where $\pi \big| [a,\, b]$ means "π is a partition of $[a,\, b]$."

Our objective is to prove some things about functions of bounded variation. One is that such functions can be written as the difference between two increasing functions. The path to proving this seems to me to require that we view the sum in (1) as a sum of positive and negative parts of the differences $f(x_i) - f(x_{i-1})$.

Here is special notation for the sums that appear in the definition of "function of bounded variation," and for two "pieces" of that sum.

$$\begin{aligned} V_f(\pi) &:= \sum_{i=1}^{n_\pi} |f(x_i) - f(x_{i-1})|, \\ P_f(\pi) &:= \sum_{i=1}^{n_\pi} (f(x_i) - f(x_{i-1}))^+, \\ N_f(\pi) &:= \sum_{i=1}^{n_\pi} (f(x_i) - f(x_{i-1}))^-. \end{aligned} \tag{2}$$

In these definitions, if x is a real number, we write $x^+ := \max(0,\, x)$ and $x^- := \max(0,\, -x)$, the *positive part* and the *negative part* of x, respectively. Then we have $|x| = x^+ + x^-$ and $x = x^+ - x^-$. We can use these last two formulas to write

$$\begin{aligned} V_f(\pi) &= P_f(\pi) + N_f(\pi) \quad \text{and} \\ f(b) - f(a) &= P_f(\pi) - N_f(\pi). \end{aligned} \tag{3}$$

The sum with no absolute value signs is telescoping. Since, for each π, $V_f(\pi) \le V_f[a,\, b]$ we have

$$P_f(\pi) + N_f(\pi) \le V_f[a,\, b].$$

Similarly,

$$V_f(\pi) \le P_f[a,\, b] + N_f[a,\, b], \tag{4}$$

where

$$P_f[a,\, b] := \sup_{\pi \big| [a,\, b]} \sum_{i=1}^{n_\pi} (f(x_i) - f(x_{i-1}))^+ \quad \text{and} \quad N_f[a,\, b] := \sup_{\pi \big| [a,\, b]} \sum_{i=1}^{n_\pi} (f(x_i) - f(x_{i-1}))^-. \tag{5}$$

The first of these is called the "positive variation," the second the "negative variation." Hence

$$V_f[a,\,b] \le P_f[a,\,b] + N_f[a,\,b]. \tag{6}$$

We can now state the Theorem we want.

Theorem: *Let f be a real-valued function defined on $[a,\,b]$. Then*

$$V_f[a,\,b] = P_f[a,\,b] + N_f[a,\,b].$$

(*BV*) *If one of $V_f[a,\,b]$, $P_f[a,\,b]$ and $N_f[a,\,b]$ is finite, then all three are finite. If f is of bounded variation,*

$$f(b) - f(a) = P_f[a,\,b] - N_f[a,\,b].$$

Proof: By (6), half of the first equation is done. To prove that the inequality in the other direction is true, we will use the observation you all made in class, that the second equation in (3) can be solved for one of the two sums in terms of the other. Therefore, for instance,

$$V_f(\pi) = P_f(\pi) + N_f(\pi) = f(b) - f(a) + 2N_f(\pi).$$

We can replace the left-hand side by its least upper bound:

$$V_f[a,\,b] \ge f(b) - f(a) + 2N_f(\pi).$$

Now we can take the supremum on the right –

$$V_f[a,\,b] \ge f(b) - f(a) + 2N_f[a,\,b].$$

Similarly,

$$V_f[a,\,b] \ge 2P_f[a,\,b] - (f(b) - f(a)).$$

On adding these two inequalities and then dividing by 2, we have

$$V_f[a,\,b] \ge P_f[a,\,b] + N_f[a,\,b]. \tag{7}$$

Hence

$$V_f[a,\,b] = P_f[a,\,b] + N_f[a,\,b].$$

We never made any assumption that any of the variations was finite. However, we *did* exploit the finiteness of each individual partition-sum. If $V_f[a,\,b]$ *is* finite, (7) shows that both of $P_f[a,\,b]$ and $N_f[a,\,b]$ are finite.

To continue, we will use the second equation in (3), and your observation, twice. Thus, from (3),

$$N_f(\pi) = P_f(\pi) - (f(b) - f(a)). \tag{8}$$

We take the supremum on the right first and then on the left. This gives

$$N_f[a,\,b] \le P_f[a,\,b] - (f(b) - f(a)). \tag{9}$$

Next we return to (8), and this time take the supremum on the left first and then on the right. This gives

$$N_f[a,\,b] \ge P_f[a,\,b] - (f(b) - f(a)). \tag{10}$$

The inequalities (9) and (10) show that, if one of $P_f[a,\,b]$ and $N_f[a,\,b]$ is finite, so is the other, and then by (6), so is $V_f[a,\,b]$. Finally, if f is of bounded variation, (9) and (10) show that the last equation in the Theorem is true. The proof is done.

If a function is of bounded variation on $[a,\,b]$ it is also of bounded variation on each subinterval $[a,\,x]$ of $[a,\,b]$. This observation leads to the Theorem known as the Jordan Decomposition Theorem, which shows that a function of bounded variation can be expressed as the difference of two increasing functions.

Theorem (Jordan Decomposition): *If f is a function of bounded variation on $[a,\,b]$ then f can be written as the difference of two increasing functions:*

$$f(x) = f(a) + P_f[a,\,x] - N_f[a,\,x].$$

Conversely, if $f(x) = g(x) - h(x)$, where g and h are increasing on $[a,\,b]$, then f is of bounded variation on $[a,\,b]$, and $V_f[a,\,b] \le V_g[a,\,b] + V_h[a,\,b]$ $(= |g(b) - g(a)| + |h(b) - h(a)|)$.

This is a Corollary of Theorem (*BV*) that we proved above..