

**Riemann-Stieltjes sums**

We suppose that  $f(x)$  and  $\alpha(x)$  are real-valued functions defined on  $[a, b]$ . In most of our applications,  $\alpha$  will be a function of bounded variation, but the *definition* does not require it!

A Riemann-Stieltjes sum for  $f$  over  $[a, b]$  with respect to the partition  $\pi$ , using the choice vector  $\xi$ , and integrator  $\alpha$ , may be denoted as follows, and it is given by the value of the sum following it:

$$RS_{\alpha}(f, [a, b], \pi, \xi) := \sum_{i=1}^{n_{\pi}} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})).$$

Let us remember that the finiteness of  $f(x)$  and the finiteness of  $\alpha(x)$ , for every  $x \in [a, b]$ , ensures that every Riemann-Stieltjes sum for  $f$  over  $[a, b]$  is finite.

Despite the absence of  $\Delta x$  we still measure the fineness of a partition using the mesh:

$$\text{mesh}(\pi) := \max_{1 \leq i \leq n_{\pi}} (x_i - x_{i-1}) = \max_{1 \leq i \leq n_{\pi}} \Delta x_i.$$

**The Riemann-Stieltjes-sum definition of the Riemann-Stieltjes integral**

**Definition:** Let  $\alpha : [a, b] \rightarrow \mathbb{R}$ . A real-valued function  $f(x)$  defined on the bounded and closed interval  $[a, b]$  is Riemann-Stieltjes integrable on  $[a, b]$  with respect to  $\alpha$  if there exists a number  $RS$  such that for all  $\epsilon > 0$  there exists  $\delta > 0$  such that for every partition  $\pi$  of  $[a, b]$ ,

$$\text{mesh}(\pi) < \delta \Rightarrow |R(f, \pi) - RS| < \epsilon.$$

We write

$$\int_a^b f(x) d\alpha(x) := RS$$

and we call this the Riemann-Stieltjes integral of  $f$  with respect to  $\alpha$  over  $[a, b]$ .

It is no longer necessary that a function  $f$  be bounded in order to be Riemann-Stieltjes integrable over  $[a, b]$  with respect to a particular  $\alpha$ . For an example, we can use  $\alpha(x) \equiv 0$ .

**The Cauchy criterion for Riemann-Stieltjes integrability**

**Theorem (Cauchy criterion for Riemann-Stieltjes integrability):** A function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann-Stieltjes integrable over  $[a, b]$  with respect to  $\alpha$  if and only if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that for all partitions  $\pi$  and  $\pi'$  of  $[a, b]$ , and for all choice vectors  $\xi$  and  $\xi'$  associated with  $\pi$  and  $\pi'$ , respectively,

$$\text{mesh}(\pi) < \delta \text{ and } \text{mesh}(\pi') < \delta \Rightarrow |R_{\alpha}(f, \pi, \xi) - R_{\alpha}(f, \pi', \xi')| < \epsilon.$$

*Proof:* First we suppose that  $f$  is Riemann-Stieltjes integrable over  $[a, b]$  with respect to  $\alpha$ . Then, using  $\epsilon/2$  in the definition of Riemann-Stieltjes integrability with respect to  $\alpha$ , we obtain  $\delta > 0$  and  $RS \in \mathbb{R}$  such that for all partitions  $\pi$  of  $[a, b]$ ,

$$\text{mesh}(\pi) < \delta \Rightarrow |R_{\alpha}(f, \pi) - RS| < \epsilon/2.$$

Now we suppose that  $\pi$  and  $\pi'$  are partitions of  $[a, b]$  and that

$$\text{mesh}(\pi) < \delta \text{ and } \text{mesh}(\pi') < \delta.$$

Then for all choice vectors  $\xi$  and  $\xi'$  associated with  $\pi$  and  $\pi'$ , respectively,

$$|R_{\alpha}(f, \pi, \xi) - R_{\alpha}(f, \pi', \xi')| \leq |R_{\alpha}(f, \pi, \xi) - RS| + |RS - R_{\alpha}(f, \pi', \xi')| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes half the proof.

Next we suppose that the Cauchy condition, given in the Theorem, is satisfied. We have to find a candidate for  $RS = \int_a^b f(x) d\alpha(x)$ . We first construct a sequence of partitions of  $[a, b]$ . We let  $\pi_n$  denote the partition that divides  $[a, b]$  into  $n$  equal parts ( $\pi_n$  has points  $x_{ni} := a + i\frac{b-a}{n}$ ). Finally we define choice vectors  $\xi_n$  by

$$\xi_{ni} := a + (i+1)\frac{b-a}{n}, \quad i = 1, \dots, n \quad \text{and define} \quad \sigma_n := \sum_{i=1}^n f(\xi_{ni})(\alpha(x_{ni}) - \alpha(x_{n,(i-1)})),$$

a Riemann-Stieltjes sum ( $\sigma_n = R_\alpha(f, \pi_n, \xi_n)$ ) with respect to  $\alpha$ . Now, given  $\epsilon > 0$ , we use  $\epsilon/2$  in the Cauchy criterion, and obtain  $\delta > 0$  such that

$$\text{mesh}(\pi) < \delta \quad \text{and} \quad \text{mesh}(\pi') < \delta \Rightarrow |R_\alpha(f, \pi, \xi) - R_\alpha(f, \pi', \xi')| < \epsilon/2.$$

Then, if  $n$  and  $n'$  are so large that  $(b-a)/n < \delta$  and  $(b-a)/n' < \delta$ , we have

$$\text{mesh}(\pi_n) < \delta \quad \text{and} \quad \text{mesh}(\pi_{n'}) < \delta \Rightarrow |\sigma_n - \sigma_{n'}| < \epsilon/2.$$

This means (since  $\epsilon$  was arbitrary) that  $\{\sigma_n\}$  is a Cauchy sequence of real numbers. Thus we define

$$RS := \lim_{n \rightarrow \infty} \sigma_n$$

and it remains to show that if  $\pi|[a, b]$  then

$$\text{mesh}(\pi) < \delta \Rightarrow |R(f, \pi) - RS| < \epsilon.$$

This is essentially done. We choose the first  $n$  such that  $\text{mesh}(\pi_n) < \delta$ , and we suppose that  $\text{mesh}(\pi) < \delta$ . Then

$$|R(f, \pi) - RS| \leq |R(f, \pi) - \sigma_n| + |\sigma_n - RS| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The proof is complete.

**The integrator and the integrand play similar rôles: integration by parts.**

It is interesting that  $f$  is Riemann-Stieltjes integrable with respect to  $\alpha$  if and only if  $\alpha$  is Riemann-Stieltjes integrable with respect to  $f$ . The proof involves the idea of summation by parts, together with a little “adding and subtracting.” This is all done at the Riemann-Stieltjes-sum level. Let’s just do it, and see later what we did.

$$\begin{aligned} RS_\alpha(f, [a, b], \pi, \xi) &= \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) \\ &= \sum_{i=1}^{n_\pi} f(\xi_i)\alpha(x_i) - \sum_{i=1}^{n_\pi} f(\xi_i)\alpha(x_{i-1}) \end{aligned}$$

and, by literal substitution of  $i+1$  for  $i$  in the second sum,

$$\begin{aligned} &= \sum_{i=1}^{n_\pi} f(\xi_i)\alpha(x_i) - \sum_{i=0}^{n_\pi-1} f(\xi_{i+1})\alpha(x_i) \\ &= - \sum_{i=1}^{n_\pi-1} (f(\xi_{i+1}) - f(\xi_i))\alpha(x_i) + f(\xi_{n_\pi})\alpha(b) - f(\xi_1)\alpha(a). \end{aligned}$$

This is almost, but not quite, what we want: a constant plus a Riemann-Stieltjes sum of  $\alpha$  with respect to  $f$ . We can subtract and add  $f(b)\alpha(b) - f(a)\alpha(a) =: f(x)\alpha(x)|_a^b$  to get what we want! Let us do this:

$$\begin{aligned}
 RS_\alpha(f, [a, b], \pi, \xi) &= - \sum_{i=1}^{n_\pi-1} (f(\xi_{i+1}) - f(\xi_i))\alpha(x_i) + f(\xi_{n_\pi})\alpha(b) - f(\xi_1)\alpha(a) \\
 &= - \sum_{i=1}^{n_\pi-1} (f(\xi_{i+1}) - f(\xi_i))\alpha(x_i) \\
 (SBP) \quad &+ (f(\xi_{n_\pi}) - f(b))\alpha(b) + (f(a) - f(\xi_1))\alpha(a) \\
 &+ f(x)\alpha(x)|_a^b \\
 &= - \sum_{i=2}^{n_\pi} (f(\xi_i) - f(\xi_{i-1}))\alpha(x_{i-1}) \\
 &- (f(\xi_1) - f(a))\alpha(a) - (f(b) - f(\xi_{n_\pi}))\alpha(b) \\
 &+ f(x)\alpha(x)|_a^b.
 \end{aligned}$$

Let's now (try to) define  $\tau$  to be the partition of  $[a, b]$  given by

$$a = \tau_0 \leq \tau_1 = \xi_1 \leq \dots \leq \tau_i = \xi_i \leq \dots \leq \tau_{n_\pi} = \xi_{n_\pi} \leq \tau_{n_\pi+1} = b.$$

At the same time, we (try to) define a choice vector  $\zeta$  by

$$\zeta_1 = a, \zeta_2 = x_1, \dots, \zeta_i = x_{i-1}, \dots, \zeta_{n_\pi} = x_{n_\pi-1}, \zeta_{n_\pi+1} = b.$$

Since  $\xi_{i-1} \leq x_{i-1} \leq \xi_i$ , we have  $\tau_{i-1} \leq \zeta_i \leq \tau_i$  for  $1 < i \leq n_\pi$ . When  $i = 1$ ,  $\tau_0 = a = \zeta_1 \leq \xi_1 = \tau_1$ . When  $i = n_\pi + 1$ , we have  $\tau_{n_\pi+1} = b = \zeta_{n_\pi+1} \geq \xi_{n_\pi} = \tau_{n_\pi}$ . Thus  $\zeta$  seems as though it is a valid choice vector for  $\tau$ .

But it is possible, using this definition, to have two consecutive  $\tau_i$  that are equal. It won't do to have consecutive points of a partition equal. This would mess up the "choose the shortest interval" tool! Suppose that  $\tau_{k-1} = \tau_k$ . Then  $\tau_{k-1} = \zeta_k = \tau_k$ . Then (claim)  $k = 1$  or  $\tau_{k-2} < \tau_{k-1}$ . In other words, at most two consecutive  $\xi_i$  can be equal, since  $x_{i-1} < x_i$  for  $1 \leq i \leq n_\pi$ . If this happens, we drop  $\tau_k$  and  $\zeta_k$  from  $\tau$  and  $\zeta$  respectively. This means that one copy of  $\xi_k$  is removed from the list that is  $\tau$ , and the  $x_{k-1}$  that is the value of  $\zeta_k$  is removed. But the other copy of  $\xi_k$ , the one that is  $\xi_{k-1} =: \tau_{k-1}$  remains in  $\tau$ . In this way we eventually arrive at a legitimate partition  $\tilde{\tau}$  of  $[a, b]$  and a valid choice vector  $\tilde{\zeta}$  associated with  $\tilde{\tau}$ . In the "most general (random)" case, the  $\xi_i$  will be interior points of the intervals of  $\pi$ . Then  $\text{mesh}(\tau) < 2 \text{mesh}(\pi)$ . There will be no duplications and  $\tau$  will have one more point than  $\pi$  does. At the other extreme,  $\tilde{\tau}$  will have about half the number of points of  $\pi$ , and  $\text{mesh}(\tilde{\tau}) \leq 2 \text{mesh}(\pi)$ .

We can rewrite (SBP) in terms of  $\tau$  and  $\zeta$ :

$$\begin{aligned}
 RS_\alpha(f, [a, b], \pi, \xi) &= - \sum_{i=2}^{n_\pi} (f(\xi_i) - f(\xi_{i-1}))\alpha(x_{i-1}) \\
 &- (f(\xi_1) - f(a))\alpha(a) - (f(b) - f(\xi_{n_\pi}))\alpha(b) \\
 (RSBP) \quad &+ f(x)\alpha(x)|_a^b. \\
 &= - \sum_{i=1}^{n_\pi+1} (f(\tau_i) - f(\tau_{i-1}))\alpha(\zeta_i) + f(x)\alpha(x)|_a^b. \\
 &= - RS_f(\alpha, [a, b], \tilde{\tau}, \tilde{\zeta}) + f(x)\alpha(x)|_a^b.
 \end{aligned}$$

In this formulation, any terms with equal  $\tau$ 's will be zero and can be eliminated by renumbering, as discussed above. The last line in (RSBP) will allow us to verify the "integration-by-parts" formula

$$(RSIBP) \quad \int_a^b f(x) d\alpha(x) = - \int_a^b \alpha(x) df(x) + f(x)\alpha(x)|_a^b.$$

In this presentation, based on the idea of starting with an idea and following it, we have arrived at a formula that has the logic backwards. Thus, given that  $\int_a^b f(x) d\alpha(x)$  exists we should have begun the last sequence of calculations with a Riemann-Stieltjes sum of  $\alpha$  with respect to  $f$ . Had we done so, the sum formula would have been

$$(df \text{ to } d\alpha) \quad RS_f(\alpha, [a, b], \pi, \xi) = -RS_\alpha(f, [a, b], \tilde{\tau}, \tilde{\zeta}) + \alpha(x)f(x)\Big|_a^b.$$

That is, beginning with a Riemann-Stieltjes sum of  $\alpha$  with respect to  $f$ , we find that it is equal to a constant minus a Riemann-Stieltjes sum of  $f$  with respect to  $\alpha$ . Moreover, we have  $\text{mesh}(\tilde{\tau}) \leq 4 \text{mesh}(\pi)$ . If we are given  $\epsilon > 0$  we now find a  $\delta$  corresponding to  $\epsilon/2$  such that

$$(*) \quad \text{mesh}(\tilde{\tau}) < \delta \Rightarrow \left| RS_\alpha(f, [a, b], \tilde{\tau}, \tilde{\zeta}) - \int_a^b f(x) d\alpha(x) \right| < \epsilon/2.$$

Next we set  $\hat{\delta} := \delta/4$ , suppose that  $\pi \ll [a, b]$  and that  $\text{mesh}(\pi) < \hat{\delta} = \delta/4$ . We can now rewrite  $(df \text{ to } d\alpha)$  as

$$(df \text{ to } d\alpha) \quad RS_f(\alpha, [a, b], \pi, \xi) - \left( -\int_a^b f(x) d\alpha(x) + f(x)\alpha(x)\Big|_a^b \right) = \int_a^b f(x) d\alpha(x) - RS_\alpha(f, [a, b], \tilde{\tau}, \tilde{\zeta}).$$

It follows from this and  $(*)$  that the integration-by-parts formula (*RSIBP*) makes sense and is valid. Since the rôles of  $\alpha$  and  $f$  can thus be reversed, it follows that  $\int f d\alpha$  exists if and only if  $\int \alpha df$  does.

### Examples

We need some examples of pairs  $(f, \alpha)$  such that  $\int f d\alpha$  exists.

**Theorem:** If  $f$  is continuous and  $\alpha$  is of bounded variation on  $[a, b]$  then  $\int_a^b f d\alpha$  exists. Moreover,

$$\left| \int_a^b f d\alpha \right| \leq \max_{[a,b]} |f| \cdot V_\alpha[a, b].$$

*Proof:* We will use an extension of the following Theorem that we proved earlier:

If  $f$  is a real-valued function defined on the interval  $[a, b]$  and  $\pi$  and  $\pi'$  are partitions of  $[a, b]$ , then

$$(1) \quad |R(f, \pi) - R(f, \pi')| \leq \sum_{i=1}^{n_\pi} \omega_i \Delta x_i + \sum_{j=1}^{n_{\pi'}} \omega'_j \Delta x'_j,$$

where  $\omega_i$  and  $\omega'_j$  denote the oscillations of  $f$  on the intervals  $I_i$  and  $I'_j$  of the respective partitions.

The extension is

**Theorem:** If  $f$  is a real-valued function defined on the interval  $[a, b]$ ,  $\pi$  and  $\pi'$  are partitions of  $[a, b]$ , and  $\alpha$  is of bounded variation on  $[a, b]$ , then

$$(2) \quad |R_\alpha(f, \pi) - R_\alpha(f, \pi')| \leq \sum_{i=1}^{n_\pi} \omega_i V_\alpha I_i + \sum_{j=1}^{n_{\pi'}} \omega'_j V_\alpha I'_j,$$

where  $\omega_i$  and  $\omega'_j$  denote the oscillations of  $f$  on the intervals  $I_i$  and  $I'_j$  of the respective partitions, and where  $V_\alpha I$  denotes the variation of  $\alpha$  over the closed interval  $I$ .

The proof of this Theorem is virtually the same as the original. The uniform continuity of  $f$  and the bounded variation of  $\alpha$  then, together with (2), give the existence of  $\int_a^b f d\alpha$ . To prove the estimate in the Theorem we begin with the absolute value of an arbitrary Riemann-Stieltjes sum of  $f$  with respect to  $\alpha$  and use the triangle inequality:

$$\left| \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) \right| \leq \sum_{i=1}^{n_\pi} |f(\xi_i)| |\alpha(x_i) - \alpha(x_{i-1})| \leq \max_{[a,b]} |f| \sum_{i=1}^{n_\pi} |\alpha(x_i) - \alpha(x_{i-1})| \leq \max_{[a,b]} |f| \cdot V_\alpha[a, b].$$

The “Dirac delta-function” arises as another example. We consider  $\alpha(x)$  defined by  $\alpha(x) = 0$  if  $x < 0$ ,  $\alpha(x) = v$  if  $x = 0$ , and  $\alpha(x) = 1$  if  $x > 0$ . Suppose that  $f$  is defined on  $[-1, 1]$  and is continuous at 0. Then, in case 0 is not a point of  $\pi$ , for some  $k$   $x_{k-1} < 0 < x_k$  and

$$RS_\alpha(f, [-1, 1], \pi, \xi) = f(\xi_k),$$

but if 0 is a point of  $\pi$ , say  $0 = x_k$ , then

$$RS_\alpha(f, [-1, 1], \pi, \xi) = f(\xi_{k-1})v + f(\xi_k)(1 - v).$$

As the mesh size of  $\pi$  tends to zero, we see that “ $d\alpha$  is zero except at 0, where it is so large that”

$$\int_{-1}^1 f d\alpha = f(0).$$

**Examples of non-existence**

**Theorem:** *If  $f$  and  $\alpha$  both have a discontinuity at  $c \in [a, b]$  then  $\int_a^b f d\alpha$  does not exist.*

*Proof:* We will use a formula for the difference between two Riemann-Stieltjes sums taken over the same partition. We can suppress  $\pi$  in the notation, and concentrate on the choice vectors. And we will use choice vectors whose components agree at all but one place.

First, suppose that  $c$  is not a point of  $\pi$ , so that for some  $k$ ,  $x_{k-1} < c < x_k$ . We define choice vectors  $\xi$  and  $\xi'$ , and choose  $\xi_i = \xi'_i$  if  $i \neq k$ . Then

$$R_\alpha(f, \xi) - R_\alpha(f, \xi') = (f(\xi_k) - f(\xi'_k))(\alpha(x_k) - \alpha(x_{k-1})).$$

Second, if  $c$  is a point of  $\pi$ , so that for some  $k$ ,  $x_{k-1} = c$ , we choose  $\xi_i = \xi'_i$  if  $i \neq k$ . Then

$$R_\alpha(f, \xi) - R_\alpha(f, \xi') = (f(\xi_k) - f(\xi'_k))(\alpha(x_k) - \alpha(c)).$$

The argument falls into cases. We will assume that  $c$  is an interior point of the interval. Only the First Case below is relevant if  $c$  is an endpoint.

**First Case:**  *$c$  is a discontinuity of both functions, on the same side.*

Let us assume that both are discontinuous from the right. This means there exists  $\epsilon > 0$  such that for all  $\delta > 0$  there exist  $x''$  and  $x'$  such that  $c < x'' < c + \delta$  and  $c < x' < c + \delta$  and  $|f(x'') - f(c)| \geq \epsilon$  and  $|\alpha(x') - \alpha(c)| \geq \epsilon$ . Now suppose that  $\text{mesh}(\pi') < \delta$ , that  $c \in \pi$ , say  $x'_{k-1} = c$ , and that  $x''$  and  $x'$ , as described just now, have been found, with the added requirement that both belong to  $[c, x'_k]$ . We now construct a partition  $\pi$  by adding  $x'$  to  $\pi$ , so that  $x_{k-1} = c$  and  $x'_k = x'$ . We select choice vectors with  $\xi_k = x''$  and  $\xi'_k = c$ . Then

$$|R_\alpha(f, \pi, \xi) - R_\alpha(f, \pi, \xi')| = |f(x'') - f(c)||\alpha(x_k) - \alpha(c)| = |f(x'') - f(c)||\alpha(x') - \alpha(c)| > \epsilon^2.$$

Since  $\delta > 0$  was arbitrary,  $\int_a^b f d\alpha$  does not exist.

**Second Case:** *We are not in the first case,  $c$  is a discontinuity of both functions, on opposite sides.*

This means that each function is continuous on one side at  $c$ . Let’s assume that  $\alpha$  is continuous on the right, and  $f$  on the left, at  $c$ . Our “bad epsilon” will once again be based on the “discontinuity epsilon” used in the First Case. However this time it will be  $\epsilon^2/2$  instead of  $\epsilon^2$ . We need an extra “delta,” say  $\delta_o > 0$ , chosen so that, for  $0 < h < \delta_o$ , we have  $|f(c - h) - f(c)| < \epsilon/2$  and  $|\alpha(c + h) - \alpha(c)| < \epsilon/2$ . Now, given a challenging  $\delta > 0$  we modify it, if necessary, so that it is not greater than  $\delta_o$ . We choose a point  $\hat{x}$ , with  $c - \delta < \hat{x} < c$ , such that  $|\alpha(\hat{x}) - \alpha(c)| \geq \epsilon$ . We choose a point  $\hat{\hat{x}}$  so that  $\hat{x} < c < \hat{\hat{x}} < \hat{x} + \delta$ . Then we construct a partition  $\pi$  of  $[a, b]$  with mesh (slightly) smaller than  $\delta$  that has  $\hat{x}$  and  $\hat{\hat{x}}$  as consecutive members, called  $x_{k-1}$  and  $x_k$ , respectively. By construction and the triangle inequality,

$$|\alpha(x_k) - \alpha(x_{k-1})| = |\alpha(x_k) - \alpha(c) + \alpha(c) - \alpha(x_{k-1})| \geq |\alpha(x_k) - \alpha(c)| - |\alpha(c) - \alpha(x_{k-1})| > \epsilon/2.$$

We can work *exactly* as in the First Case for  $f$ , except that now  $c < x'' < \hat{\hat{x}}$ . We get

$$|R_\alpha(f, \pi, \xi) - R_\alpha(f, \pi, \xi')| = |f(x'') - f(c)||\alpha(x_k) - \alpha(x_{k-1})| > \epsilon^2/2.$$

Since  $\delta > 0$  was arbitrary,  $\int_a^b f d\alpha$  does not exist.