The Axioms for the Real Numbers, a Complete, Ordered, Field

(0) There exists a set $\mathbb{R}$, and two operations, called *addition,* and *multiplication,* denoted $x+y$, xy, (or $x \cdot y$) respectively, that have the following properties: (text's **A–1** and **M–1**)

(1) $x+(y+z)=(x+y)+z$ for all x, y, z in $\mathbb{R}$; addition is *associative;* (text's **A–3**)

(2) $x+y=y+x$ for all x, y, z in $\mathbb{R}$ addition is *commutative;* (**A–2**)

(3) there exists an element z in $\mathbb{R}$ such that $x+z=x$ for every x in $\mathbb{R}$; (part of **A–4**)

(4) for each x in $\mathbb{R}$, there exists w in $\mathbb{R}$ such that $x+w=z$; (part of **A–5**)

(5) $x(yz)=(xy)z$ for all x, y, z in $\mathbb{R}$; multiplication is *associative;* (**M–3**)

(6) $xy=yx$ for all x, y in $\mathbb{R}$; multiplication is *commutative;* (**M–2**)

(7) there exists an element u in $\mathbb{R}$ such that $u \cdot x = x$ for every x in $\mathbb{R}$; (part of **M–4**)a

(8) if $x \neq 0$, there exists $v \in \mathbb{R}$ such that $x \cdot v = u$. (part of **M–5**)

(9) $x(y+z)=xy+xz$ for all x, y, z in $\mathbb{R}$; multiplication is *distributive* over addition; (**D**)

(10) $u \neq z$. (**M–4**)b

As axioms, the statements to this point are called *the axioms for a field.*

(Order Properties) There exists a subset $\mathbb{R}^+$ of $\mathbb{R}$ (the *positive real numbers)* (**I**)a
such that

(11) for all x in $\mathbb{R}$, exactly one of the statements $x=z$, $x \in \mathbb{R}^+$, and $-x \in \mathbb{R}^+$ is true; (**I**)b

(12) if $x \in \mathbb{R}^+$, and $y \in \mathbb{R}^+$, then $x+y \in \mathbb{R}^+$, and $xy \in \mathbb{R}^+$. (**I**)c

As axioms, the statements to this point are called *the axioms for an ordered field.*

(13) **(Completeness Axiom)** For all subsets S of $\mathbb{R}$, if S is non-empty and bounded above, then there exists a unique element of $\mathbb{R}$, denoted $\sup(S)$, such that

(i) $\sup(S)$ is an upper bound for S, and

(ii) for all x in $\mathbb{R}$, if $x < \sup(S)$ then there exists $s \in S$ such that $x < s$.
(**Theorem (3.4);** not actually (**C**), but equivalent to it)

Here, $\sup(S)$ is called the *supremum* of the set S; this is also known as the *least upper bound* of the set S, denoted $\text{lub}(S)$. We will use "supremum" and "sup." Please note that the axiom "says" that $\sup(S)$ is the smallest upper bound for S; the axiom also "says" that the set of all upper bounds of a non-empty set that is bounded above has a *least* element!

As axioms, the statements to this point are called *the axioms for a complete ordered field.*
There is essentially only *one* complete ordered field. There are *many* different ordered fields.

Related Theorems, Definitions and Notation

Theorems: *The elements called z and u in Axioms (3) and (7) are unique; they are denoted 0 and 1, respectively. Given x in Axiom (4) or $x \neq 0$ in Axiom (8), the corresponding elements w and v are unique. They are called, respectively, the additive inverse of x, denoted $-x$, and the multiplicative inverse of (a non–zero) x, denoted x^{-1}.*

Definition: *If $x \in \mathbb{R}$ and $y \in \mathbb{R}$ then x is less than y, denoted $x < y$, if $y - x \in \mathbb{R}^+$.*

We write $x \leq y$ if the statement $(x=y) \vee (x<y)$ is true.

Definition: *If $S \subseteq \mathbb{R}$ then S is bounded above if there exists $M \in \mathbb{R}$ such that $x \leq M$ for every $x \in S$.*