

**The Axioms for the Real Numbers**, a Complete, Ordered, Field

(0) There exists a set  $\mathbb{R}$ , and two operations, called *addition*, and *multiplication*, denoted  $x + y$ ,  $xy$ , (or  $x \cdot y$ ) respectively, that have the following properties: (text's **A-1** and **M-1**)

- (1)  $x + (y + z) = (x + y) + z$  for all  $x, y, z$  in  $\mathbb{R}$ ; addition is *associative*; (text's **A-3**)
- (2)  $x + y = y + x$  for all  $x, y, z$  in  $\mathbb{R}$  addition is *commutative*; (**A-2**)
- (3) there exists an element  $z$  in  $\mathbb{R}$  such that  $x + z = x$  for every  $x$  in  $\mathbb{R}$ ; (part of **A-4**)
- (4) for each  $x$  in  $\mathbb{R}$ , there exists  $w$  in  $\mathbb{R}$  such that  $x + w = z$ ; (part of **A-5**)
- (5)  $x(yz) = (xy)z$  for all  $x, y, z$  in  $\mathbb{R}$ ; multiplication is *associative*; (**M-3**)
- (6)  $xy = yx$  for all  $x, y$  in  $\mathbb{R}$ ; multiplication is *commutative*; (**M-2**)
- (7) there exists an element  $u$  in  $\mathbb{R}$  such that  $u \cdot x = x$  for every  $x$  in  $\mathbb{R}$ ; (part of **M-4**)a
- (8) if  $x \neq 0$ , there exists  $v \in \mathbb{R}$  such that  $x \cdot v = u$ . (part of **M-5**)
- (9)  $x(y + z) = xy + xz$  for all  $x, y, z$  in  $\mathbb{R}$ ; multiplication is *distributive* over addition; (**D**)
- (10)  $u \neq z$ . (**M-4**)b

As axioms, the statements to this point are called *the axioms for a field*.

**(Order Properties)** There exists a subset  $\mathbb{R}^+$  of  $\mathbb{R}$  (the *positive real numbers*) (**I**)a such that

- (11) for all  $x$  in  $\mathbb{R}$ , exactly one of the statements  $x = z$ ,  $x \in \mathbb{R}^+$ , and  $-x \in \mathbb{R}^+$  is true; (**I**)b
- (12) if  $x \in \mathbb{R}^+$ , and  $y \in \mathbb{R}^+$ , then  $x + y \in \mathbb{R}^+$ , and  $xy \in \mathbb{R}^+$ . (**I**)c

As axioms, the statements to this point are called *the axioms for an ordered field*.

(13) **(Completeness Axiom)** For all subsets  $S$  of  $\mathbb{R}$ , if  $S$  is non-empty and bounded above, then there exists a unique element of  $\mathbb{R}$ , denoted  $\sup(S)$ , such that

- (i)  $\sup(S)$  is an upper bound for  $S$ , and
- (ii) for all  $x$  in  $\mathbb{R}$ , if  $x < \sup(S)$  then there exists  $s \in S$  such that  $x < s$ .  
(**Theorem (3.4)**; not actually (**C**), but equivalent to it)

Here,  $\sup(S)$  is called the *supremum* of the set  $S$ ; this is also known as the *least upper bound* of the set  $S$ , denoted  $\text{lub}(S)$ . We will use “supremum” and “sup.” Please note that the axiom “says” that  $\sup(S)$  is the smallest upper bound for  $S$ ; the axiom also “says” that the set of all upper bounds of a non-empty set that is bounded above has a *least* element!

As axioms, the statements to this point are called *the axioms for a complete ordered field*.

There is essentially only one complete ordered field. There are many different ordered fields.

**Related Theorems, Definitions and Notation**

**Theorems:** The elements called  $z$  and  $u$  in Axioms (3) and (7) are unique; they are denoted  $0$  and  $1$ , respectively. Given  $x$  in Axiom (4) or  $x \neq 0$  in Axiom (8), the corresponding elements  $w$  and  $v$  are unique. They are called, respectively, the additive inverse of  $x$ , denoted  $-x$ , and the multiplicative inverse of (a non-zero)  $x$ , denoted  $x^{-1}$ .

**Definition:** If  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  then  $x$  is less than  $y$ , denoted  $x < y$ , if  $y - x \in \mathbb{R}^+$ .

We write  $x \leq y$  if the statement  $(x = y) \vee (x < y)$  is true.

**Definition:** If  $S \subseteq \mathbb{R}$  then  $S$  is bounded above if there exists  $M \in \mathbb{R}$  such that  $x \leq M$  for every  $x \in S$ .