

The Axioms for the Real Numbers, a Complete, Ordered, Field

(0) There exists a set \mathbb{R} , and two operations, called *addition*, and *multiplication*, denoted $x + y$, xy , (or $x \cdot y$) respectively, that have the following properties:

- (1) $x + (y + z) = (x + y) + z$ for all x, y, z in \mathbb{R} ; addition is *associative*;
- (2) $x + y = y + x$ for all x, y, z in \mathbb{R} addition is *commutative*;
- (3) there exists an element z in \mathbb{R} such that $x + z = x$ for every x in \mathbb{R} ;
- (4) for each x in \mathbb{R} , there exists w in \mathbb{R} such that $x + w = z$;
- (5) $x(yz) = (xy)z$ for all x, y, z in \mathbb{R} ; multiplication is *associative*;
- (6) $xy = yx$ for all x, y in \mathbb{R} ; multiplication is *commutative*;
- (7) there exists an element u in \mathbb{R} such that $u \cdot x = x$ for every x in \mathbb{R} ;
- (8) if $x \neq 0$, there exists $v \in \mathbb{R}$ such that $x \cdot v = u$.
- (9) $x(y + z) = xy + xz$ for all x, y, z in \mathbb{R} ; multiplication is *distributive* over addition;
- (10) $u \neq z$.

As axioms, the statements to this point are called *the axioms for a field*.

(Order Properties) There exists a subset \mathbb{R}^+ of \mathbb{R} (the *positive real numbers*) such that

- (11) for all x in \mathbb{R} , exactly one of the statements $x = z$, $x \in \mathbb{R}^+$, and $-x \in \mathbb{R}^+$ is true;
- (12) if $x \in \mathbb{R}^+$, and $y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$, and $xy \in \mathbb{R}^+$.

As axioms, the statements to this point are called *the axioms for an ordered field*.

(13) **(Completeness Axiom)** For all subsets S of \mathbb{R} , if S is non-empty and bounded above, then there exists a unique element of \mathbb{R} , denoted $\sup(S)$, such that

- (i) $\sup(S)$ is an upper bound for S , and
- (ii) for all x in \mathbb{R} , if $x < \sup(S)$ then there exists $s \in S$ such that $x < s$.

Here, $\sup(S)$ is called the *supremum* of the set S ; this is also known as the *least upper bound* of the set S , denoted $\text{lub}(S)$. We will use “supremum” and “sup.” Please note that the axiom “says” that $\sup(S)$ is the smallest upper bound for S ; the axiom also “says” that the set of all upper bounds of a non-empty set that is bounded above has a *least* element!

As axioms, the statements to this point are called *the axioms for a complete ordered field*.

There is essentially only *one* complete ordered field. There are *many* different ordered fields.

Related Theorems, Definitions and Notation

Theorems: The elements called z and u in Axioms (3) and (7) are unique; they are denoted 0 and 1 , respectively. Given x in Axiom (4) or $x \neq 0$ in Axiom (8), the corresponding elements w and v are unique. They are called, respectively, the additive inverse of x , denoted $-x$, and the multiplicative inverse of (a non-zero) x , denoted x^{-1} or $1/x$ or $\frac{1}{x}$.

Definition: If $x \in \mathbb{R}$ and $y \in \mathbb{R}$ then x is less than y , denoted $x < y$, if $y - x \in \mathbb{R}^+$.

We write $x \leq y$ if the statement $(x = y) \vee (x < y)$ is true.

Definition: If $S \subseteq \mathbb{R}$ then S is bounded above if there exists $M \in \mathbb{R}$ such that $x \leq M$ for every $x \in S$.