

The Axioms for a Complete Ordered Field

(0) There exists a set \mathbb{R} , and two operations, called *addition*, and *multiplication*, denoted $x + y$, xy , (or $x \cdot y$) respectively, that have the following properties:

- (1) $x + (y + z) = (x + y) + z$ for all x, y, z in \mathbb{R} ; addition is *associative*;
- (2) $x + y = y + x$ for all x, y, z in \mathbb{R} addition is *commutative*;
- (3) there exists a unique *additive identity*, denoted 0 , such that $x + 0 = x$ for each x in \mathbb{R} ;
- (4) for each x in \mathbb{R} , there exists w in \mathbb{R} such that $x + w = 0$; w is called an *additive inverse* of x and is denoted $-x$;
- (5) $x(yz) = (xy)z$ for all x, y, z in \mathbb{R} ; multiplication is *associative*;
- (6) $xy = yx$ for all x, y in \mathbb{R} ; multiplication is *commutative*;
- (7) there exists a unique *multiplicative identity*, denoted 1 , such that $1 \cdot x = x$ for each x in \mathbb{R} ;
- (8) if $x \neq 0$, there exists $v \in \mathbb{R}$ such that $x \cdot v = 1$; v is called a *multiplicative inverse* of x , denoted $1/x$ or x^{-1} ;
- (9) $x(y + z) = xy + xz$ for all x, y, z in \mathbb{R} ; multiplication is *distributive* over addition;
- (10) $1 \neq 0$.

As axioms, the statements to this point are called *the axioms for a field*.

(Order Properties) There exists a subset \mathbb{R}^+ of \mathbb{R} such that,

- (11) for all x in \mathbb{R} , exactly one of the statements $x = 0$, $x \in \mathbb{R}^+$, and $-x \in \mathbb{R}^+$ is true;
- (12) if $x \in \mathbb{R}^+$, and $y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$, and $xy \in \mathbb{R}^+$.

As axioms, the statements to this point are called *the axioms for an ordered field*.

(13) (**Completeness Property**) For all subsets S of \mathbb{R} , if S is non-empty and bounded above, then there exists a unique element, denoted $\sup(S)$, in \mathbb{R} such that

- (i) $\sup(S)$ is an upper bound for S ,

and

- (ii) for all x in \mathbb{R} , if $x < \sup(S)$ then there exists $s \in S$ such that $x < s$.

As axioms, the statements to this point are called *the axioms for a complete ordered field*.