The Recursion Theorem: its use justifies "definition by induction"

Introduction (may be skipped) The Recursion Theorem is an Existence and Uniqueness Theorem. It asserts the existence of a certain kind of sequence in a specified non-empty set X.

Let's look at an example: Let $x_1 := 1$, and decree that for each $n \geq 1$, $x_{n+1} = \frac{1}{2}\left(\frac{2}{x_n} + x_n\right)$. This is a "definition by induction," also known as "a recursive sequence." If you are bothered by this, you are absolutely right! Does this process *really* define a sequence? The answer is that it does, but a Theorem is needed to connect $\mathbb{N}$ and the set theory that is needed to define functions.

We need to know exactly what a "function" is! A function is "actually" a set of ordered pairs, namely what we usually call the *graph* of the function. In the example, the set of ordered pairs is not actually specified, as they are in the definition $F_n := \frac{n(n+1)}{2}$. The recursive definition tells us how to do the next step, so we know that for any finite set of arguments we can construct ordered pairs, but it does not justify saying that the infinite set of ordered pairs needed to define the function actually exists! The Recursion Theorem justifies this recursive definition by asserting that such a function indeed exists.

Definition: *Let X and Y be non-empty sets. A function f with domain X and range (space) Y is a subset $f \subseteq X \times Y$ that has the following properties:*

(1) For all $x \in X$ there exists $y \in Y$ such that $(x, y) \in f$
(2) For all $x \in X$, for all $y_1 \in Y$ and for all $y_2 \in Y$, if $(x, y_1) \in f$ and $(x, y_1) \in f$ then $y_1 = y_2$.

We might call (1) "the rule" and (2) "the vertical line test." The standard notation expressing that f is a function f with domain X and range Y is: $f : X \to Y$. The set f is actually the graph of the function, and we usually write $y = f(x)$ instead of saying that $(x, y) \in f$. Among many synonyms is "f maps X to Y." The graph of the example $F_n := \frac{n(n+1)}{2}$ consists of all the ordered pairs $\left(n, \frac{n(n+1)}{2}\right)$, one for each $n \in \mathbb{N}$.

We next state the Recursion Theorem in "function" form. We'll state the "sequence" version too, in the Remarks.

The Recursion Theorem *Let Y be a non-empty set, and suppose that $x_0 \in Y$. Suppose also that $H : Y \to Y$ is a function. Then there exists a unique function $R : \mathbb{N} \to Y$ such that $R(0) = x_0$ and such that for all $n \in \mathbb{N}$, $R(n+1) = H(R(n))$.*

Remarks: The first version of the proof to follow is very leisurely. It is based on the proof given in the book *Naive Set Theory,* by Paul Halmos. A shorter version will follow. You might want to go directly to the short version! In the statement of the Theorem we could have written $x_n := R(n)$, and this would give us "x_0 is given, and for each n, $x_{n+1} = H(x_n)$." The Recursion Theorem (stated for $\mathbb{P} := \mathbb{N} \setminus \{0\}$ instead of $\mathbb{N}$) justifies the definition of the sequence given in the Introduction: Y is the set of positive real numbers, with $x_1 = 1$ and $H(x) = \frac{1}{2}\left(\frac{2}{x} + x\right)$. The proof has features that may remind you of our discussion of Inductive Sets.

Proof of existence: Our objective is to construct a set R of ordered pairs that has properties (1) and (2) of a function $R : \mathbb{N} \to Y$. We will begin by using set–selector notation to select unimaginably many sets of ordered pairs! We will define a set $\mathcal{F}_0$ of subsets of $\mathbb{N} \times Y$. We often use synonyms, such as "family," for the word "set" in this context so we won't have to say things like "let $\mathcal{F}_0$ be the following set of sets..." We define the family $\mathcal{F}_0 \subseteq 2^{\mathbb{N} \times Y}$ by

$$\mathcal{F}_0 := \{S \in 2^{\mathbb{N} \times Y} : (0, x_0) \in S \text{ and } (\forall n \in \mathbb{N})(\forall x \in Y)((n, x) \in S \Rightarrow (n+1, H(x)) \in S)\}.$$

In words (to be read *very* slowly), $\mathcal{F}_0$ is the family of all sets S of ordered pairs in $\mathbb{N} \times Y$ such that $(0, x_0) \in S$ and, whenever $(n, x) \in S$, then $(n+1, H(x)) \in S$ as well. Since $\mathbb{N} \times Y$ contains all possible ordered pairs of natural numbers n and elements x of Y, $\mathbb{N} \times Y \in \mathcal{F}_0$. Thus $\mathcal{F}_0$ is a non-empty family of non-empty sets S of ordered pairs. Each S is required to have two properties, as specified in the definition of $\mathcal{F}_0$.

The set R of ordered pairs we want is the set of all ordered pairs that are in every one of the sets $S \in \mathcal{F}_0$. Before we form this set, let's check that it can "act" like a function. We define

$$E_0 := \{n \in \mathbb{N} : \text{ there exists } x \in Y \text{ such that } (n, x) \in S \text{ for all } S \in \mathcal{F}_0\}.$$

Let us show that every natural number n is in E_0. This will show that when we form the set R of all ordered pairs in $\mathbb{N} \times Y$ that are in every $S \in \mathcal{F}_0$ (we have a notation for this: $R = \bigcap_{S \in \mathcal{F}_0} S$), it will be true that for every $n \in \mathbb{N}$ there exists $x \in Y$ such that $(n, x) \in R$. This is the requirement (1) that R must satisfy if R is to be a function from $\mathbb{N}$ to Y.

Proof that $E_0 = \mathbb{N}$: We will use induction. By definition, $(0, x_0) \in S$ for every $S \in \mathcal{F}_0$. Thus $0 \in E_0$. Now we suppose that $n \in E_0$. This means that there is some $x \in Y$ such that (n, x) belongs to every $S \in \mathcal{F}_0$. But $(n, x) \in S \in \mathcal{F}_0$ means that $(n+1, H(x)) \in S \in \mathcal{F}_0$. Therefore, for every $S \in \mathcal{F}_0$, $(n+1, H(x)) \in S$, so $n+1 \in E_0$. That is, $0 \in E_0$ and $n \in E_0 \Rightarrow n+1 \in E_0$, so $n \in E_0$ for all n, so $E_0 = \mathbb{N}$.

Now we officially define the set R:

$$R := \bigcap_{S \in \mathcal{F}_0} S, \quad \text{the set of all ordered pairs } (n, x) \text{ that are in every } S \in \mathcal{F}_0.$$

(3) **Exercise:** *Prove that* $R \in \mathcal{F}_0$.
(4) **Exercise:** *Prove that* R *is the smallest set in* $\mathcal{F}_0$.

To prove that R is (the graph of) a function with domain $\mathbb{N}$ and range Y, we have to verify two things:

(i) For every $n \in \mathbb{N}$ there exists $x \in Y$ such that $(n, x) \in R$;
(ii) For all n in $\mathbb{N}$ and all x_1 and x_2 in Y, if $(n, x_1) \in R$ and $(n, x_2) \in R$ then $x_1 = x_2$.

We already showed that (i) is true for R, because for every $n \in \mathbb{N}$, there is an $x \in Y$ such that $(n, x) \in S$ for every $S \in \mathcal{F}_0$. Therefore $(n, x) \in R$.

To verify (ii) we might begin by supposing that $(n, x_1) \in R$ and $(n, x_2) \in R$. We would then have to show that $x_1 = x_2$. We will do this, but by induction. We define

$$E_1 := \big\{ n \in \mathbb{N} : (\forall x_1 \in Y)(\forall x_2 \in Y)\big(\big[(n, x_1) \in R \wedge (n, x_2) \in R\big] \Rightarrow \big[x_1 = x_2\big]\big)\big\}.$$

If we can show that $E_1 = \mathbb{N}$, this will show, for arbitrary $n \in \mathbb{N}$ and arbitrary x_1 and x_2 in Y, that whenever $(n, x_1) \in R$ and $(n, x_2) \in R$ then $x_1 = x_2$.

Proof that $E_1 = \mathbb{N}$: We will use this idea: assume the contrary, and then construct sets in $\mathcal{F}_0$ that are *strictly smaller* than R. Since R is the smallest set in $\mathcal{F}_0$, this will give the desired contradictions.

By construction, $(0, x_0) \in R$. If also $(0, y_0) \in R$ and $y_0 \neq x_0$, we construct a new set $S' := R \setminus \{(0, y_0)\}$. This set has to be in $\mathcal{F}_0$. Firstly, $(0, x_0) \in R$, and, since $x_0 \neq y_0$, $(0, x_0) \in R \setminus \{(0, y_0)\} = S'$ (i.e., when we took $(0, y_0)$ out of R, we did not take out $(0, x_0)$). Secondly, if $(m, x) \in S'$ then $(m, x) \in R$, so $(m+1, H(x)) \in R$. Moreover, $(m+1, H(x)) \neq (0, y_0)$ because $m+1 \neq 0$ for any m, and for ordered pairs to be unequal it's enough if one pair of two corresponding coordinates is unequal. Therefore $(m+1, H(x)) \in R \setminus \{(0, y_0)\} = S'$. Therefore $S' \in \mathcal{F}_0$. But S' is R with one element removed, so S' is strictly smaller than the smallest set in $\mathcal{F}_0$. This is a contradiction, so $0 \in E_1$.

We assume $n \in E_1$, and seek to show that this implies $n+1 \in E_1$. Again we proceed by contradiction, assuming that $n+1 \notin E_1$. Since $n \in E_1$, $(n, x) \in R$ for a *unique* $x \in Y$. Then $(n+1, H(x)) \in R$. However, by our assumption of the contrary, there is also some $y_1 \in Y$ such that $y_1 \neq H(x)$ and $(n+1, y_1) \in R$ as well. We construct another set, $S'' := R \setminus \{(n+1, y_1)\}$.

Our strategy again is to show that $S'' \in \mathcal{F}_0$. As before, this will give a contradiction.

Once again, $(0, x_0) \in S''$ because $0 \neq n+1$ for any natural number means that $(0, x_0) \neq (n+1, y_1)$.

Now we suppose that $(m, y) \in S''$. There are two cases to consider: $m \neq n$ and $m = n$.

If $m \neq n$, we have $m+1 \neq n+1$, so $(m+1, H(y)) \neq (n+1, y_1)$.
However, $(m+1, H(y)) \in R$ because $(m, y) \in S'' \subseteq R$.
And $(m+1, H(y)) \neq (n+1, y_1)$, so $(m+1, H(y)) \in R \setminus \{(n+1, y_1)\} = S''$.

That is, when $m \neq n$, $(m, y) \in S'' \Rightarrow (m+1, H(y)) \in S''$.

Finally we have to deal with the case $m = n$. Here, $(m, y) = (n, y) \in R$, so $y = x$ (by uniqueness, because $n \in E_1$) and therefore $(m+1, H(y)) = (n+1, H(x)) \in R$. Since $y_1 \neq H(x)$, $(m+1, H(y)) = (n+1, H(x)) \neq (n+1, y_1)$. Hence $(m+1, H(y)) = (n+1, H(x)) \in S''$.

Thus $S'' \in \mathcal{F}_0$, which gives a contradiction. Thus $n+1 \in E_1$, so $E_1 = \mathbb{N}$. Hence (ii) is true for R so R is a function.

Proof of uniqueness: Suppose R and R' satisfy the conditions of the Theorem. That is, $R(0) = x_0 = R'(0)$ and for all $n \in \mathbb{N}$, $R(n+1) = H(R(n))$ and $R'(n+1) = H(R'(n))$. We define the set

$$E_2 := \{n \in \mathbb{N} : R(n) = R'(n)\}.$$

To show that $R' = R$ all we have to do is show that $E_2 = \mathbb{N}$. We use induction. We are given that $0 \in E_2$. If $n \in E_2$, then $R'(n) = R(n)$. But then $R'(n+1) = H(R'(n)) = H(R(n)) = R(n+1)$. The second equality is by Substitution. The first and third equalities are given. Thus $n+1 \in E_2$, so $E_2 = \mathbb{N}$ and so $R' = R$.

A shorter version of the proof We define the family $\mathcal{F}_0 \subseteq 2^{\mathbb{N} \times Y}$ by

$$\mathcal{F}_0 := \{S \in 2^{\mathbb{N} \times Y} : (0, x_0) \in S \text{ and } (\forall n \in \mathbb{N})(\forall x \in Y)((n, x) \in S \Rightarrow (n+1, H(x)) \in S)\}.$$

In words, $\mathcal{F}_0$ is the family of all sets S of ordered pairs in $\mathbb{N} \times Y$ such that $(0, x_0) \in S$ and, whenever $(n, x) \in S$, then $(n+1, H(x)) \in S$ as well. Since $\mathbb{N} \times Y$ contains all possible ordered pairs of natural numbers n and elements x of Y, $\mathbb{N} \times Y \in \mathcal{F}_0$.

The set R of ordered pairs we want is the set of all ordered pairs that are in every one of the sets $S \in \mathcal{F}_0$:

$$R := \bigcap_{S \in \mathcal{F}_0} S, \quad \text{the set of all ordered pairs } (n, x) \text{ that are in every } S \in \mathcal{F}_0.$$

To prove that R is (the graph of) a function with domain $\mathbb{N}$ and range Y, we have to verify two things:

(i) For every $n \in \mathbb{N}$ there exists $x \in Y$ such that $(n, x) \in R$;
(ii) For all n in $\mathbb{N}$ and all x_1 and x_2 in Y, if $(n, x_1) \in R$ and $(n, x_2) \in R$ then $x_1 = x_2$.

To verify (i) we define

$$E_0 := \{n \in \mathbb{N} : \text{ there exists } x \in Y \text{ such that } (n, x) \in S \text{ for all } S \in \mathcal{F}_0\}.$$

Let us show that every natural number n is in E_0. We will use induction. By definition, $(0, x_0) \in S$ for every $S \in \mathcal{F}_0$. Thus $0 \in E_0$. Next we suppose that $n \in E_0$. Now $(n, x) \in S \in \mathcal{F}_0$ means that $(n+1, H(x)) \in S \in \mathcal{F}_0$. Therefore, for every $S \in \mathcal{F}_0$, $(n+1, H(x)) \in S$, so $n+1 \in E_0$. But then $E_0 = \mathbb{N}$ and thus (i) is true for R.

To verify (ii) we define

$$E_1 := \big\{n \in \mathbb{N} : (\forall x_1 \in Y)(\forall x_2 \in Y)\big(\big[(n, x_1) \in R \wedge (n, x_2) \in R\big] \Rightarrow \big[x_1 = x_2\big]\big)\big\}.$$

If we can show that $E_1 = \mathbb{N}$, this will show, for arbitrary $n \in \mathbb{N}$ and arbitrary x_1 and x_2 in Y, that whenever $(n, x_1) \in R$ and $(n, x_2) \in R$ then $x_1 = x_2$.

Proof that $E_1 = \mathbb{N}$: We will use this idea: assume the contrary, and then construct sets in $\mathcal{F}_0$ that are *strictly smaller* than R. Since R is the smallest set in $\mathcal{F}_0$, this will give the desired contradictions.

By construction, $(0, x_0) \in R$. If also $(0, y_0) \in R$ and $y_0 \neq x_0$, we construct $S' := R \setminus \{(0, y_0)\}$. We show $S' \in \mathcal{F}_0$. Firstly, $(0, x_0) \in R$, and, since $x_0 \neq y_0$, $(0, x_0) \in R \setminus \{(0, y_0)\} = S'$. Secondly, if $(m, x) \in S'$ then $(m, x) \in R$, so $(m+1, H(x)) \in R$. Moreover, $(m+1, H(x)) \neq (0, y_0)$ because $m+1 \neq 0$ for any m. Therefore $(m+1, H(x)) \in R \setminus \{(0, y_0)\} = S'$. Therefore $S' \in \mathcal{F}_0$. But S' is strictly smaller than R, the smallest set in $\mathcal{F}_0$. This is a contradiction, so $0 \in E_1$.

We assume $n \in E_1$, and seek to show that this implies $n+1 \in E_1$. Again we proceed by contradiction, assuming that $n+1 \notin E_1$. Since $n \in E_1$, $(n, x) \in R$ for a *unique* $x \in Y$. Then $(n+1, H(x)) \in R$. However, by our assumtion of the contrary, there is also some $y_1 \in Y$ such that $y_1 \neq H(x)$ and $(n+1, y_1) \in R$ as well. We construct another set, $S'' := R \setminus \{(n+1, y_1)\}$.

Our strategy again is to show that $S'' \in \mathcal{F}_0$. As before, this will give a contradiction.
Since $0 \neq n+1$ for any natural number, $(0, x_0) \in S''$.

We suppose $(m, y) \in S''$ and consider two cases: $m \neq n$ and $m = n$.

If $m \neq n$, $m+1 \neq n+1$, so $(m+1, H(y)) \neq (n+1, y_1)$.
As $(m+1, H(y)) \neq (n+1, y_1)$, $(m+1, H(y)) \in R \setminus \{(n+1, y_1)\} = S''$.
That is, when $m \neq n$, $(m, y) \in S'' \Rightarrow (m+1, H(y)) \in S''$.

If $m = n$, $(m, y) = (n, y)$ so $y = x$ and therefore $(m+1, H(y)) = (n+1, H(x)) \in R$. Since $H(x) \neq y_1$, we have $(m+1, H(y)) \in S''$.

Thus $S'' \in \mathcal{F}_0$, which gives a contradiction. Hence (ii) is true for R so R is a function.

To prove uniqueness suppose R and R' satisfy the conditions of the Theorem. We define the set

$$E_2 := \{n \in \mathbb{N} : R(n) = R'(n)\}.$$

To show that $R' = R$ we show by induction that $E_2 = \mathbb{N}$. We are given that $0 \in E_2$. If $n \in E_2$, then $R'(n) = R(n)$. But then $R'(n+1) = H(R'(n)) = H(R(n)) = R(n+1)$. Thus $n+1 \in E_2$, so $E_2 = \mathbb{N}$ and so $R' = R$.

This completes the proof of the Recursion Theorem.

Example: Let us modify Rudin's proof of **3.7** so that it uses the Recursion Theorem. The proof starts off murkily, because if E^* consisted of just one point, it would have no limit points. Thus we assume that E^* has limit points, for otherwise every point of E^* would be isolated, and then E^* would be closed, as desired. The *idea* is to look for new points p_n in smaller and smaller balls $B_r(q)$, and we want to make a subsequence that way, so we have to find subscripts on the p_n that increase strictly.

Let $T := \{n \in \mathbb{N} : p_n \neq q\}$; then $T \neq \emptyset$, and we let $n_1 := \min T$. We set $\delta := d(q, p_{n_1})$. Then $B_{\delta/2}(q)$ contains points $x \in E^*$ such that $x \neq q$ (why?). Hence $B_{\delta/2}(q)$ contains terms p_n (close to x, for example). Indeed, there are infinitely many *terms* of $\{p_n\}$ in $B_{\delta/2}(q)$, though maybe not infinitely many *points* from $\{p_n\}$ in $B_{\delta/2}(q)$.

Let us try to create a suitable set Y that works for the *idea* of the proof, stated above.

We note that, for each i, the set $T_i := \{n \in \mathbb{P} : p_n \in B_{\delta/2^i}(q)\}$ is infinite because q is a limit point of E^*, so that every open set containing a point of E^* also contains a term p_n from $\{p_n\}$ (why?). Our problem is to say how to select a term $p_{n_i} \in T_i$ in such a way that $n_{i+1} > n_i$. Let's use n_1 to start. We can use a set of integers that does not contain any integer less than the ones we have already selected. Let's try

$$Y := \{(S_n, n) \in 2^{\mathbb{P}} \times \mathbb{P} : S_n := \{m \in \mathbb{P} : m > n \text{ and } p_m \in B_{\delta/2^n}(q)\}\}.$$

By what we noted at the start of the last paragraph, each S_n that we defined is non-empty. In fact, each S_n depends entirely on n, so Y is an infinite set. Our function? Let's try

$$H(S_n, n) := (S_N, N), \quad \text{where } N := \min S_n.$$

Since every $m \in S_n$ is strictly greater than n, $N > n$. This rule gives a function defined on Y that takes its values in Y. Our starting point will be (S_{n_1}, n_1).

The Recursion Theorem gives us a sequence $y_i := (S_{n_i}, n_i)$ of "points" of Y such that $y_1 = (S_{n_1}, n_1)$ and such that $y_{i+1} = H(S_{n_i}, n_i)$ for all $i \in \mathbb{P}$.

Then $n_i \geq i$ and $p_{n_i} \in B_{\delta/2^{n_i}}(q) \subseteq B_{\delta/2^i}(q)$, so $d(p_{n_i}, q) < 1/2^i$, so $p_{n_i} \to q$, and thus $q \in E^*$, as desired.