The Replacement Lemma: *Let* $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_M\}$ *be a finite subset of a vector space* V *such that* $\operatorname{span} S = V$ *and suppose that no* $s_j = \mathbf{0}$ *Let* $L = \{\mathbf{v}_1, \ldots, \mathbf{v}_K\}$ *be a linearly independent subset of* V. *Then there is a subset* S_1 *of* S *having* K *elements such that* $\operatorname{span}\{(S \setminus S_1) \cup L\} = V$. *In particular,* $K \le M$.

Sketch of a proof: Since $\operatorname{span} S = V$ there exist coefficients a_{1j}, $j = 1, \ldots, M$ such that

$$\text{(1)} \qquad \mathbf{v}_1 = \sum_{j=1}^{M} a_{1j}\mathbf{s}_j, \quad \text{and there is a } \textit{first} \ j, \ \text{say } J_1, \ \text{such that} \quad a_{1J_1} \ne 0.$$

Otherwise, $\mathbf{v}_1$ would be zero, but $\mathbf{v}_1 \ne 0$ because $\mathbf{v}_1$ is a member of a linearly independent set. Thus

$$\text{(2)} \qquad -a_{1J_1}\mathbf{s}_{J_1} = \sum_{\substack{j=1 \\ j \ne J_1}}^{M} a_{1j}\mathbf{s}_j - \mathbf{v}_1, \quad \text{or} \quad \mathbf{s}_{J_1} = -\frac{1}{a_{1J_1}} \sum_{\substack{j=1 \\ j \ne J_1}}^{M} a_{1j}\mathbf{s}_j + \frac{1}{a_{1J_1}}\mathbf{v}_1.$$

The expression on the far right in (2) is a linear combination of the vectors in $(S \setminus \{\mathbf{s}_{J_1}\}) \cup \{\mathbf{v}_1\}$. Hence every linear combination of the vectors in S can be re-written as a linear combination of the vectors in $(S \setminus \{\mathbf{s}_{J_1}\}) \cup \{\mathbf{v}_1\}$. Conversely, because of (1) every linear combination of the vectors in $(S \setminus \{\mathbf{s}_{J_1}\}) \cup \{\mathbf{v}_1\}$ can be re-written as a linear combination of the vectors in S. Thus the two sets S and $(S \setminus \{\mathbf{s}_{J_1}\}) \cup \{\mathbf{v}_1\}$ have the same span, which is V.

Next we replace L by $L_1 := \{\mathbf{v}_2, \ldots, \mathbf{v}_K\}$. If this set is non-empty we can repeat the argument we just did, with, however, S replaced by $(S \setminus \{\mathbf{s}_{J_1}\}) \cup \{\mathbf{v}_1\}$. We can then write

$$\mathbf{v}_2 = \sum_{\substack{j=1 \\ j \ne J_1}}^{M} a_{2j}\mathbf{s}_j + b_{21}\mathbf{v}_1, \quad \text{and there is a } \textit{first} \ j \ne J_1, \ \text{say } J_2, \ \text{such that} \quad a_{2J_2} \ne 0.$$

Otherwise, we would have $\mathbf{v}_2 = b_{21}\mathbf{v}_1$ and this is impossible because L is a linearly independent set. We can thus "replace" $\mathbf{s}_{J_2}$ by $\mathbf{v}_2$ in the same way we "replaced" $\mathbf{s}_{J_1}$ by $\mathbf{v}_1$. Thus $\operatorname{span}\{(S \setminus \{\mathbf{s}_{J_1}, \mathbf{s}_{J_2}\}) \cup \{\mathbf{v}_1, \mathbf{v}_2\}\} = V$.

We can continue removing one vector at a time from L, "replacing" a vector from the remaining elements of S because there will always be a non-zero coefficient on one of the remaining $\mathbf{s}_j$'s since otherwise the $\mathbf{v}_k$ we're working with would be a linear combination of the other vectors in L.

In this way, we can remove all the $\mathbf{v}_k$'s from L. Thus $K \le M$ because there have to be at least K vectors in S if they can all be "replaced!" We let S_1 denote the set of $\mathbf{s}_j$'s we removed from S, and this gives the statement in the Lemma.

An application of the Replacement Lemma

Corollary: *Any two bases of the same finite-dimensional vector space have the same number of elements.*

Proof: Let the two bases take turns being S and L in the Replacement Lemma.