Second version; these will be revised and added to
Practice! Practice! Practice! Good Luck!

1 State the Principle of Mathematical Induction, and prove by induction that

$$(\forall n \in \mathbb{N})(n > 1 \Rightarrow n^2 > n + 1).$$

Here, $n^2 := n \cdot n$; the algebraic properties of addition and multiplication can be used freely.

2 Define *null sequence.* Prove that $\{1/n^2\}$ is a null sequence. You are given that $\{1/n\}$ is a null sequence.

3 Prove that the equation $r^2 = 3$ has no solution $r \in \mathbb{Q}$

4 Let X be a set. Define the *complement,* S^c, of a subset S of X. Prove that, if A and B are subsets of X, then: $A \subseteq B$ if and only if $B^c \subseteq A^c$.

5 In the context of the Peano Postulates, define "less than or equal to." Supply the appropriate quantifiers and prove that if $m \leq n$ and $n \leq m$ then $m = n$.

6 Construct a wide truth table that: (a) shows $A \Rightarrow B$ and its contrapositive are logically equivalent but that $A \Rightarrow B$ and its converse are not; (b) shows whether or not, if C denotes the converse of $A \Rightarrow B$, the statement $(A \Rightarrow B) \vee C$ is a tautology.

7 State the Trichotomy Law for the natural numbers. Explain why $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(m < n \Rightarrow m^2 < n^2)$ is TRUE but $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x < y \Rightarrow x^2 < y^2)$ is FALSE. No proofs are required!

8 Prove that for all $\nu \in \mathbb{Z}^+$ and all positive $p \in \mathbb{R}$, $(1 + p)^\nu > \nu p$. Hint: look at $(1+p)^{\nu+1} - (1+p)^\nu$ first. You do not have to prove that $(1+p)^\nu > 1$.

9 Write the following statement "in logic," using logic symbols and our symbolic notation for the set names:

"Between any two distinct positive real numbers there is a natural number."

Explain why the statement is false.

10 State the Completeness Axiom. Prove that, if S is a non-empty set of real numbers and r is a real number such that

(a) r is an upper bound for S
and (b) $r \in S$,

then $r = \sup S$.

11 Prove that if $\{n_k\}$ is a strictly increasing sequence of natural numbers then $n_k \geq k$ for all natural numbers k (this shows that $n_k \to \infty$).

12 Prove that the Harmonic Series diverges. Do not use Cauchy Condensation.

13 State Cauchy's Condensation Test. Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ for convergence.

14 State the Theorem relating absolute convergence and convergence. Give an example of a series that is convergent (why convergent?) but not absolutely convergent.

15 If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge to the same value, is it true that a_n and b_n converge to the same value? Justify your answer!

16 Test for convergence (identify any convergence tests that you use):

(a) $\displaystyle\sum_{r=0}^{\infty} \frac{1}{r!}$; (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$, $x > 0$; (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^n}$.

17 What is a Geometric Series? What are the partial sums of a Geometric Series? Which Geometric Series converge? Why?

18 State the limit versions of the Root Test and the Ratio Test. Test $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ $(x > 0)$ for convergence.

19 *Prove* that $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges if $\{b_n\}$ is a convergent sequence, by calculating the partial sums. Find the sum of the series. You are proving that telescoping series converge.

20 Prove that, if $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ both converge for $|z| < R$, then for every pair c, d of complex numbers, $\sum_{n=0}^{\infty}(ca_n + db_n)z^n$ converges for $|z| < R$.

21 Let y be a positive real number. Prove that there exists one and only one $k \in \mathbb{Z}$ such that $2^k \le y < 2^{k+1}$.

22 If we write $\log y$ to denote x, use the Chain Rule and the equation $e^{\log y} = y$ to find the derivative of $\log y$ with respect to y, *assuming* that $\log y$ is differentiable.

23 Show that $\log e^x = x$ for all real x.

24 Show that $\log y_1 y_2 = \log y_1 + \log y_2$ for all positive numbers y_1 and y_2. What is $\log 1$? $\log e$? $\log(1/y)$, in terms of $\log y$?

25 Suppose that $\sum z_n$ is a series with complex terms, $z_n = x_n + iy_n$, where x_n and y_n are real numbers. Prove that $\sum z_n$ converges if and only if $\sum x_n$ and $\sum y_n$ converge.

26 **Telescoping Series** Find s_N for each series. If the series converges, find its sum. If the series diverges, say why.

(a) $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ (b) $\sum_{n=1}^{\infty} \frac{n}{(n+2)(n+3)}$ (c) $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+4}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n^2+(m+1)n+m}$, where $1 < m \in \mathbb{Z}^+$.

27 Test the following series for convergence. Justify your answers! If x appears in the series, $x > 0$ is assumed.

(a) $\sum_{n=2}^{\infty} \binom{n}{2} x^n$; (b) $\sum_{n=m}^{\infty} \binom{n}{m} x^n$, $m \in \mathbb{Z}^+$; (c) $\sum_{n=1}^{\infty} \frac{x^n}{n}$; (d) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$; (e) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$; (f) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; (g) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$.

28 Test the following series for convergence. Justify your answers!

(a) $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2-4}}$ (b) $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^3-8}}$ (c) $\sum_{n=3}^{\infty} \frac{n}{\sqrt{n^3-8}}$ (d) $\sum_{n=3}^{\infty} \sqrt{\frac{n}{n^3-8}}$ (e) $\sum_{n=3}^{\infty} \sqrt{\frac{n}{n^4-16}}$ (f) $\sum_{n=3}^{\infty} \sqrt{\frac{n^2}{n^5-32}}$

29 Let $x > 0$. Test the following series for convergence. Justify your answers! Use # 4 for part of each answer.

(a) $\sum_{n=3}^{\infty} \frac{x^n}{\sqrt{n^2-4}}$ (b) $\sum_{n=3}^{\infty} \frac{x^n}{\sqrt{n^3-8}}$ (c) $\sum_{n=3}^{\infty} \frac{nx^n}{\sqrt{n^3-8}}$ (d) $\sum_{n=3}^{\infty} \sqrt{\frac{nx^n}{n^3-8}}$ (e) $\sum_{n=3}^{\infty} \sqrt{\frac{nx^n}{n^4-16}}$ (f) $\sum_{n=3}^{\infty} \sqrt{\frac{n^2x^n}{n^5-32}}$

30 Test $\sum_{n=1}^{\infty} a_n$ for convergence where, in each case,

(a) $a_{2n} = 1/4^n$, $a_{2n+1} = 1/5^n$; (b) $a_n = \frac{1}{kn}$, where $2^{k-1} \le n < 2^k$ so the first few a_n are $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}, \frac{1}{15}$.

31 Test for convergence. Justify your answers! Here, x is arbitrary.

(a) $\sum_{n=0}^{\infty}(-1)^n$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ (c) $\sum_{n=0}^{\infty} x^n$ (d) $\sum_{n=1}^{\infty} nx^n$ (e) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (f) $\sum_{n=1}^{\infty} e^n \frac{x^n}{\sqrt{n}}$ (g) $\sum_{n=1}^{\infty} \frac{x^{n^2}}{n}$

32 Here the coefficients a_n are the ones in # 6 and x is arbitrary. Test each $\sum a_n x^n$ for convergence.

33 Prove that if $\sum z_n$ and $\sum w_n$ are convergent series with complex terms then

(a) $\sum(z_n + w_n)$ converges; (b) $\sum(z_n - w_n)$ converges. (c) Must $\sum z_n w_n$ converge?

34 Suppose that $\{s_n\}$ converges. Prove that the series $\sum_{n=1}^{\infty}(s_{n+1} - s_n)$ converges.

35 Differentiate the series $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ term-by-term. What do you get?

Do the same for the series $L(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. What do you get?

36 Suppose that $y_1 > 0$ and $y_2 > 0$, and that $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$. Are x_1 and x_2 unique? Find x_3 and x_4 so that $y_1 y_2 = e^{x_3}$ and $y_1/y_2 = e^{x_4}$. What are $e^{x_3+x_4}$ and $e^{x_3-x_4}$?

37 Suppose that $y > 0$ and that $y = e^x$. Find the n-th root of y in terms of x.

38 Given: e^x is continuous, $e^0 = 1$ and $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. Find (and prove, using what was given)

$$\lim_{x\to 0} \frac{e^{2x} - e^x}{x} \left(= \lim_{x\to 0} \frac{e^{2x} - e^x}{2x - x} \right) \text{ and } \lim_{x\to 0} \frac{e^{x^2} - e^x}{x^2 - x}.$$

39 Using that $(e^x)' = e^x$ and Calculus, prove that for x positive and n a positive integer, $g_n(x) := x^n e^{-x}$ increases at first as x increases from 0, then decreases. Find the point x_n at which the change from increasing to

decreasing occurs, and find the limit, as $x \to \infty$, of $g_n(x)$. Do not use l'Hospital's rule unless you prove it. Hint: you can get a useful upper bound for $g_n(x)$ by replacing e^x by $f_{n+1}(x)$. The Binomial Theorem is also handy here, along with the Squeeze Theorem.

40 Prove that, if $x \neq y$ are real numbers, then $e^{(x+y)/2} < \dfrac{e^x + e^y}{2}$.

41 Suppose that n is an arbitrary positive integer such that $n > 2$.
Prove that, for $0 \le k \le (n-2)/2$, it is true that $\binom{n}{k} < \binom{n}{k+1}$.

42 Prove that $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ is a decreasing sequence.

43 Calculate $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^k$, $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$ and $\displaystyle\sum_{k=0}^{n} \binom{n}{k} (-2)^k$.

44 Prove that the series $E(x) := \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} \left(= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}, \text{ by definition!}\right)$ converges for each $x > 0$.

46 Given that $x > 1$, prove that there is a positive integer K such that $\dfrac{x^k}{k!}$ increases for $0 \le k \le K$, then decreases strictly for $k > K$. Find K for $x = 2$, 4 and 8. Find $\displaystyle\lim_{k\to\infty} \frac{x^k}{k!}$. Justify your answer!

47 Calculate $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{3^{k+1}}$, $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{4^{k+1}}$ and, for $m \in \mathbb{Z}^+$, $\displaystyle\sum_{k=0}^{\infty} \frac{m^k}{(m+1)^{k+1}}$.

48 Suppose $z = x + iy \neq 0$. Find the real and imaginary parts of $1/z$.

49 Suppose $z = x + iy$, where $x > y > 0$. Plot z and z^2 in the plane, as line segments from 0 to the *points* z and z^2. Use analytic geometry methods or trigonometry to show that the angle between z^2 and the positive x–axis is twice the angle between z and the positive x–axis.

50 Plot the points that are in the sequence $\{i^n\}$.

51 Find all *complex* solutions z of the equation $z^2 = i$.

52 Find all *complex* solutions z of the equation $z^3 = 1$.

53 Find all *complex* solutions z of the equation $z^4 = 1$.

54 Plot all *complex* solutions z of the equation $z^5 = 1$. Don't seek formulas for them. Explain you method.

55 Are the Difference–of–Powers Formula and the Binomial Theorem true for complex numbers? Explain why.

56 Prove that multiplication of complex numbers is commutative and associative.

57 Prove that if $x_n \to L$ and $x_n \to M$ then $M = L$. This is called "uniqueness of limits."

58 Prove that $\{(-1)^n\}$ diverges.

59 Prove that for positive m and n, and all real x, that $(x^m)^n = x^{mn}$. Do the same if they are simply integers and $x \neq 0$. The rule $x^{-n} = 1/x^n$ when $x \neq 0$ is one you brought with you. It is actually a definition!

60 Prove that if $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ and $x \neq 0$ then $(x^m)^n = (x^n)^m$.

61 Prove that if $|x| < 1$ then $\{n^2 x^n\}$ is a null sequence.

62 Prove that if $|x| \ge 1$ then $\{n^3 x^n\}$ is **not** a null sequence.

63 Prove that if $|x| < 1$ then $\{n^3 x^n\}$ is a null sequence.

64 Prove that, if $m \in \mathbb{N}$ and $n \in \mathbb{N}$ and $m \le n/2$ then $\dbinom{n}{m} > \dfrac{n^m}{2^m m!}$.

65 Prove that if $|x| < 1$ and $m \in \mathbb{Z}^+$ then $\{n^m x^n\}$ is a null sequence.

66 Give two proofs that if $0 < s < t$ then $s^n < t^n$ for all $n \in \mathbb{Z}^+$, one by induction and one using the Difference-of-Powers Formula (DoPF).

67 Use the DoPF to prove that, if $s > 0$ and $t > 0$ then for all $n \in \mathbb{Z}^+$, $s < t \iff s^n < t^n$.

68 Prove that $\{\frac{n+1}{n}\}$ is a strictly decreasing sequence, and that $\{\frac{n}{n+1}\}$ is a strictly increasing sequence.

Note: To prove that a sequence $\{x_n\}$ of positive numbers is strictly increasing, there are two main methods: (1) prove that $x_n/x_{n+1} < 1$; (2) prove that $x_{n+1} - x_n > 0$. Method (2) can be used even if the x_n's are not all positive.

69 Prove that if $x > 1$ then $\{x^n\}$ is a strictly increasing sequence.

70 Prove that if $x > 1$ then $\{x^n/n\}$ is **eventually** a strictly increasing sequence.

We keep in mind that $\{1/n\}$ is a null sequence.

71 Prove that $\{1/n^2\}$ is a null sequence.
72 Prove that $\{1/2^n\}$ is a null sequence.
73 Prove that for all natural numbers n, $2^n > n$.
74 Prove that, if $c \in \mathbb{R}$ and $\{x_n\}$ is a null sequence, then $\{cx_n\}$ and $\{x_n^2\}$ are null sequences.
75 Prove that for all real numbers a and b, $|ab| \le \dfrac{a^2+b^2}{2}$.
76 Prove that, if $\{x_n\}$ and $\{y_n\}$ are null sequences, so are $\{x_n + y_n\}$ and $\{x_n y_n\}$.
77 Prove that if $\{x_n\}$ is a null sequence and $\{y_n\}$ is a sequence such that $|y_n| \le |x_n|$ for all n then $\{y_n\}$ is a null sequence.
78 Prove that for all $\nu \in \mathbb{Z}^+$ and all positive $p \in \mathbb{R}$, $(1+p)^\nu > \nu p$.
79 Suppose that $x \in \mathbb{R}$, and that for every $\epsilon > 0$, $|x| < \epsilon$. Prove that $x = 0$.
80 Give an example of a set X and two functions $f : X \to X$ and $g : X \to X$ such that $f \circ g \ne g \circ f$.
81 How do we define what it means for two functions to be equal?
82 Suppose that X, Y and Z are non–empty sets and that $F : X \to Y$, $G : Y \to Z$. Prove that, if each function is one–to–one, so is their composite function, __ $\circ$ __.
83 Suppose that X, Y and Z are non–empty sets and that $F : X \to Y$, $G : Y \to Z$. Prove that, if each function is onto, so is their composite function, __ $\circ$ __.
84 Suppose that S and T are subsets of a "universal" set X. We say that S *is contained in* T if every element of S is an element of T, and we denote this by $S \subseteq T$. Use set–selector notation to *prove* that for *any* subsets S and T of X, it is true that $S \cap T \subseteq S$.
85 Suppose that S and T are subsets of a "universal" set X. What can be said about T if $T = S \cup T$? Why?
86 Suppose that S and T are subsets of a "universal" set X. What can be said about T if $T = S \cap T$? Why?
87 Suppose that X is a "universal" set. What does $(\forall S \in 2^X)$ mean?
88 Let $X := \{a, b, c\}$ and let $Y := \{d, e, f\}$. How many elements does $X \times Y$ have? How many elements does $2^{X \times Y}$ have?
89 If X is a set that is a universe of discourse, we define the *complement* of a set S to be the set of all elements of X that are not in S. We denote the complement of S by S^c. Use set–selector notation to express S^c. Prove that $(S^c)^c = S$.
90 Prove that $(S \cup T)^c = S^c \cap T^c$ and that $(S \cap T)^c = S^c \cup T^c$. These are versions of DeMorgan's Laws.

91 If $A \Rightarrow B$ is true, and B is FALSE, what does this tell us about A?
92 Write down the Truth table for $(\sim B \Rightarrow \sim A) \iff (A \Rightarrow B)$.
$\sim B \Rightarrow \sim A$ is called the *contrapositive* of $A \Rightarrow B$.
93 Write down the Truth table for $(B \Rightarrow A) \iff (A \Rightarrow B)$.
$B \Rightarrow A$ is called the *converse* of $A \Rightarrow B$.
94 Suppose $(A \wedge B) \Rightarrow C$ is FALSE. What does this tell us about the truth values of A and B and C?
95 Are $(A \wedge B) \Rightarrow C$ and $A \wedge (B \Rightarrow C)$ the same logically? Why?
96 How many different Truth Tables for A and B are possible? Why?
97 Suppose $(A \vee B) \Rightarrow C$ is TRUE, and C FALSE. What does this tell us about the truth values of A and B?

98 State and prove the Power Fact (about positive integer powers of positive numbers)

99 Prove that, if $\{z_n\}$ converges, then $\{z_n - z_{n+1}\}$ is a null sequence. Give an example of a sequence $\{z_n\}$ such that $z_n - z_{n+1} \to 0$, but such that $\{z_n\}$ does not have a limit. Your sequence does not have to be complex!

100 Prove that a complex sequence $\{z_n\}$ converges to z if and only if its sequences of real and imaginary parts converge to $\mathbf{Re}\, z$ and $\mathbf{Im}\, z$, respectively.

101 Suppose that the sequence defined recursively by $x_1 := 1$ and $x_{n+1} := \dfrac{1}{1+x_n}$ converges. What is the limit, assuming it exists?

102 Prove that, if $\{z_n\}$ and $\{z'_n\}$ are sequences with limits z and z' respectively, then for all complex numbers a and b, the sequence $\{az_n + bz'_n\}$ has limit $az + bz'$. Give an "epsilon-N" proof.

103 State the Theorem about the convergence of decreasing sequences.
Use the Theorem to prove: $0 < x < 1 \Rightarrow \lim_{n\to\infty} x^n = 0$.

104 Suppose that A is a non-empty subset of $\mathbb{R}$, that $B \subseteq \mathbb{R}$ is bounded above, and suppose that $A \subseteq B$. Prove that $\sup A$ and $\sup B$ both exist, and that $\sup A \leq \sup B$.

105 Prove that the sequence $\{(-1)^n\}$ has no limit as $n \to \infty$.

106 Prove that $\lim_{n\to\infty} \dfrac{1}{2^n} = 0$.

107 Prove that, if S is a bounded non-empty set of real numbers and $\sup S \notin S$, then there exists a sequence $\{s_n\}$ such that $s_n \in S$ for all natural numbers n, and $\lim_{n\to\infty} s_n = \sup S$. Hint: Consider the numbers $\sup S - (1/n)$.

108 State and prove the Difference of Powers Formula. Suggestion: use a literal substitution.

109 Prove that, if $\{x_n\}$ is an increasing sequence of real numbers that is bounded above, then $\{x_n\}$ converges.

110 What are the distinct numbers in the sequence $\{i^n\}$?
111 Find the real and imaginary parts of $z := \frac{1-i}{1+i}$.
11 Calculate $w_n := \sum_{k=0}^{n-1} z^k$ and find the distinct numbers in the sequence $\{w_n\}$, where $z := \frac{1-i}{1+i}$.

112 Let $x_1 = 1$, $x_2 = \frac{1}{1+1}$, $x_3 = \frac{1}{1+\frac{1}{1+1}}$, $x_4 = \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}$, and so on. What is the limit, assuming it exists?

113 Prove that, if $\{z_n\}$ and $\{z'_n\}$ are sequences with limits z and z' respectively, then the sequence $\{z_n z'_n\}$ has limit zz'. Give an "epsilon-N" proof, assuming that $|z_n| \leq M$ and $|z'_n| \leq M'$. You may take it for granted that $|z| \leq M$ and $|z'| \leq M'$. Suggestion: subtract and add!

114 Prove that, for all positive integers , if $0 \leq r \leq n$ then $\rho_{nr} := \left(\frac{n+1}{n}\right)^r \frac{n+1-r}{n+1} \leq 1$. Suggestion: calculate ρ_{n0}, then calculate $\rho_{nr} - \rho_{n,r+1}$.

115 Suppose that A is a non-empty subset of $\mathbb{R}$, that $B \subseteq \mathbb{R}$ is bounded above, and suppose that $A \subseteq B$. Prove that $\sup A$ and $\sup B$ both exist, and that $\sup A \leq \sup B$.

116 Let $x_1 := 1/10$, and $x_{n+1} = 1 - x_n$, for $n \geq 1$. Prove that $\{x_n\}$ has no limit as $n \to \infty$.

117 State the n–th root Theorem. Show that if $x > 1$ then $x^{1/n} \to 1$ as $n \to \infty$. You may use without proof that $x^{1/n} > 1$ if $x > 1$.

118 Prove that, if S is a bounded non-empty set of real numbers and $\sup S \notin S$, then there exists a sequence $\{s_n\}$ such that $s_n \in S$ for all natural numbers n, and $\lim_{n\to\infty} s_n = \sup S$. Hint: Consider the numbers $\sup S - (1/n)$.