Introduction

We will approach the Riemann integral using Riemann sums instead of the upper and lower sums you may be more familiar with. The main reasons are that I want to study Riemann-Stieltjes integrals with "integrators" $\alpha(x)$ that are not monotone, but are "of bounded variation," and (most important) I want to define Riemann integrals when the values of my functions belong to an infinite dimensional vector space, where upper and lower sums don't make sense. This makes little difference in the case of real-valued functions, since (as we will see later) functions of bounded variation can always be expressed as the difference of two monotone functions.

Most of this note will be devoted to definitions and notation. I will include two major thorems as well, one that Riemann integrable functions *must* be bounded, and the other a Cauchy condition for the existence of the Riemann integral of a bounded function defined on a bounded and closed interval.

Riemann sums

A *Riemann sum* for a function $f(x)$ defined on an interval $[a, b]$ is formed with the help of

(1) A *partition* π of $[a, b]$, namely an ordered, finite set of points x_i, with $a = x_0 < x_1 < \cdots < x_n = b$ (where n is a positive integer that can be any positive integer, and one that we will often write as $n = n_\pi$)
and
(2) A *choice vector* $\xi = (\xi_1, \ldots, \xi_n)$ that has n_π components that must satisfy $x_{i-1} \le \xi_i \le x_i$, for $i = 1, 2, \ldots, n$.

Then a *Riemann sum for* f *over* $[a, b]$ *with respect to the partition* π, *using the choice vector* ξ, may be denoted (tho it seldom is) as follows, and it is given by the value of the sum following it:

$$R(f, [a, b], \pi, \xi) := \sum_{i=1}^{n_\pi} f(\xi_i)(x_i - x_{i-1}).$$

It will usually be enough to write $R(f, \pi, \xi)$ or even $R(f, \pi)$ to stand for a Riemann sum.

Riemann-Stieltjes sums

Later, we will work with *Riemann-Stieltjes sums,* and these involve a function $\alpha(x)$ that takes the place of x in a Riemann sum. Thus we suppose $\alpha(x)$ is a real-valued function defined on $[a, b]$. In our applications, α will usually be a *function of bounded variation,* to be defined later. We can call $\alpha(x)$ an *integrator.*

A *Riemann-Stieltjes sum for* f *over* $[a, b]$ *with respect to the partition* π, *using the choice vector* ξ, *and integrator* α, may be denoted as follows, and it is given by the value of the sum following it:

$$RS_\alpha(f, [a, b], \pi, \xi) := \sum_{i=1}^{n_\pi} f(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})).$$

More notation; the mesh (size) of a partition

In both of these definitions we can write $\Delta x_i := x_i - x_{i-1}$ or $\Delta\alpha_i := \alpha(x_i) - \alpha(x_{i-1})$. These are convenient because they are short and suggest the dx or $d\alpha$ in an integral. But they can cause confusion because they leave out the dependence they have on x_{i-1}.

A partition π can be thought of as "dividing" the interval $[a, b]$ into subintervals. We may write $\pi \big| [a, b]$ and read this as "π divides $[a, b]$," or "partitions $[a, b]$." We will denote the *intervals of* π by $I_i := [x_{i-1}, x_i]$ When we wish to work with 2 partitions at the same time we will often use π' to denote the second one, and then use x'_j to denote its points and I'_j to denote its intervals. We might use n' in place of $n_{\pi'}$ to denote the number of points in π'.

We measure the fineness of a partition (this is a crude measure!) using the length of the longest interval in the partition. This number is written

$$\text{mesh}(\pi) := \max_{1 \le i \le n_\pi} (x_i - x_{i-1}) = \max_{1 \le i \le n_\pi} \Delta x_i.$$

This definition of mesh size is used even in the context of Riemann-Stieltjes sums!

The Riemann-sum definition of the Riemann integral

Definition: *A real-valued function* $f(x)$ *defined on the bounded and closed interval* $[a, b]$ *is Riemann integrable on* $[a, b]$ *if there exists a number* RI *such that for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for every partition* π *of* $[a, b]$,

$$mesh(\pi) < \delta \Rightarrow |R(f, \pi) - RI| < \epsilon.$$

We write

$$\int_a^b f(x)\, dx := RI$$

and we call this the Riemann integral of f *over* $[a, b]$.

If we imagine the *set* of all numbers $R(f, \pi)$ that can be formed (using all possible appropriate choice vectors) and all possible partitions whose mesh sizes are less than δ, the definition demands that they all lie in the open interval $(RI - \epsilon, RI + \epsilon)$. This leads to a Theorem.

Theorem: *If* f *is Riemann integrable on* $[a, b]$ *then* f *is bounded on* $[a, b]$.

Proof: Suppose not. Then there exists a sequence $\{x_n\}$ in $[a, b]$ such that $|f(x_n)| > n$. Since $f(x)$ is finite at every point x in $[a, b]$, there are infinitely many distinct x_n, and so some subsequence converges to a point x^* in $[a, b]$. We now choose $\epsilon = 1$ in the definition, and obtain a corresponding $\delta > 0$. We can then construct a partition π with mesh size less than δ in such a way that x^* is contained in the *interior* of some interval I_{i_o} of π, unless x^* is an endpoint of $[a, b]$. In that case, we can still use the following argument, with $I_{i_o} = I_1$ or $I_{i_o} = I_{n_\pi}$. We now know that I_{i_o} contains infinitely many of the x_n. We next pick the components ξ_i of a choice vector ξ in an arbitrary way when $i \neq i_o$, and we let ξ_{i_o} be some $x_N \in I_{i_o}$. Then $|R(f, \pi, \xi) - RI| < 1$. We next modify ξ by changing only $\xi_{i_o} = x_N$ to $\xi'_{i_o} := x_M$, where $x_M \in I_{i_o}$, and we call the new choice vector ξ'. Then

$$R(f, \pi, \xi') - RI = R(f, \pi, \xi) - RI + (f(x_M) - f(x_N))\Delta x_{i_o}.$$

By choosing M very large compared to N we can arrange it so that $|f(x_M) - f(x_N)|\Delta x_{i_o} > 2$.

If $(f(x_M) - f(x_N))\Delta x_{i_o} > 0$, then

$$R(f, \pi, \xi') - RI > R(f, \pi, \xi) - RI + 2 > -1 + 2 = 1.$$

We argue similarly if $(f(x_M) - f(x_N))\Delta x_{i_o} < 0$. The definition is contradicted. Hence f is bounded if it is Riemann integrable.

A difficulty with the definition; the Cauchy criterion for Riemann integrability

In order to tell whether f is Riemann integrable we have to know $\int_a^b f(x)\, dx$. The idea of a Cauchy sequence leads to the following Theorem, which gives an equivalent definition.

Theorem (Cauchy criterion for Riemann integrability): *A bounded function* $f : [a, b] \to \mathbb{R}$ *is Riemann integrable over* $[a, b]$ *if and only if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all partitions* π *and* π' *of* $[a, b]$, *and for all choice vectors* ξ *and* ξ' *associated with* π *and* π', *respectively,*

$$mesh(\pi) < \delta \text{ and } mesh(\pi') < \delta \Rightarrow |R(f, \pi, \xi) - R(f, \pi', \xi')| < \epsilon.$$

Proof: First we suppose that f is Riemann integrable over $[a, b]$. Then, using $\epsilon/2$ in the definition of Riemann integrability, we obtain $\delta > 0$ and $RI \in \mathbb{R}$ such that for all partitions π of $[a, b]$,

$$\text{mesh}(\pi) < \delta \Rightarrow |R(f, \pi) - RI| < \epsilon/2.$$

Now we suppose that π and π' are partitions of $[a, b]$ and that

$$\text{mesh}(\pi) < \delta \text{ and } \text{mesh}(\pi') < \delta.$$

Then for all choice vectors ξ and ξ' associated with π and π', respectively,

$$|R(f,\pi,\xi) - R(f,\pi',\xi')| \le |R(f,\pi,\xi) - RI| + |RI - R(f,\pi',\xi')| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes half the proof.

Next we suppose that the Cauchy condition, given in the Theorem, is satisfied. We have to find a candidate for $\int_a^b f(x)\,dx$. We first construct a sequence of partitions of $[a, b]$. We let π_n denote the partition that divides $[a, b]$ into n equal parts $\left(\pi_n \text{ has points } x_{ni} := a + i\dfrac{b-a}{n}\right)$. Finally we define choice vectors ξ_n by

$$\xi_{ni} := a + (i+1)\frac{b-a}{n}\ ,\ i = 1, \ldots, n \text{ and define } \sigma_n := \sum_{i=1}^{n} f(\xi_{ni})\frac{b-a}{n},$$

a Riemann sum $(\sigma_n = R(f, \pi_n, \xi_n))$. Now, given $\epsilon > 0$, we use $\epsilon/2$ in the Cauchy criterion, and obtain $\delta > 0$ such that

$$\mathrm{mesh}(\pi) < \delta \text{ and } \mathrm{mesh}(\pi') < \delta \Rightarrow |R(f,\pi,\xi) - R(f,\pi',\xi')| < \epsilon/2.$$

Then, if n and n' are so large that $(b-a)/n < \delta$ and $(b-a)/n' < \delta$, we have

$$\mathrm{mesh}(\pi_n) < \delta \text{ and } \mathrm{mesh}(\pi_{n'}) < \delta \Rightarrow |\sigma_n - \sigma_{n'}| < \epsilon/2.$$

This means (since ϵ was arbitrary) that $\{\sigma_n\}$ is a Cauchy sequence of real numbers. Thus we define

$$RI := \lim_{n\to\infty} \sigma_n$$

and it remains to show that if $\pi\big|[a, b]$ then

$$\mathrm{mesh}(\pi) < \delta \Rightarrow |R(f,\pi) - RI| < \epsilon.$$

This is essentially done. We choose the first n such that $\mathrm{mesh}(\pi_n) < \delta$, and we suppose that $\mathrm{mesh}(\pi) < \delta$. Then

$$|R(f,\pi) - RI| \le |R(f,\pi) - \sigma_n| + |\sigma_n - RI| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The proof is complete.