All our conventions for this proof are still in force.

To show: $E \in \Sigma_F$ if and only if $E \in \Sigma$ and $\mu(E) < \infty$.

If $E \in \Sigma_F$ then every $K \cap E \in \Sigma_F$ because Σ_F is closed under intersections. Thus $E \in \Sigma_F$.

Let $\epsilon > 0$ be given. Suppose that $E \in \Sigma$ and $\mu(E) < \infty$. There exist $V \supseteq E$ such that $\mu(V) < \mu(E) + \epsilon$ and $K \subseteq V$ such that $\mu(V \setminus K) < \epsilon$ (by the Lemma). Now $E \cap K \in \Sigma_F$, so there exists a compact $L \subseteq E \cap K$ such that $\mu(E \cap K) < \mu(L) + \epsilon$. Then $E = (E \cap K) \cup (E \setminus K) \subseteq (E \cap K) \cup (V \setminus K)$ so $\mu(E) \leq \mu(L) + 2\epsilon$. Hence $\mu(E) = \inf\{\mu(L) : L \text{ is compact and } L \subseteq E\}$ so $E \in \Sigma_F$.

To show: μ is a measure on Σ. Suppose $\{E_k\}$ is a disjoint sequence in Σ. Let $E := \bigcup_k E_k$. If $\mu(E) = \infty$, $\sum_k \mu(E_k) = \infty$ by subadditivity. If $\mu(E) < \infty$ then each $\mu(E_k) < \infty$ by monotonicity. Thus if in addition to disjointness we have that $\mu(E) < \infty$ and each $\mu(E_k) < \infty$ then by the step just done our sets are all in Σ_F and the additivity has already been shown in that case.

To show: $\ell(g) = \int_X g\, d\mu$ for all $g \in C_c(X)$.

First, suppose that $g \geq 0$. We let $K := \operatorname{supp} g$, set $G := \max_x g(x)$ and $0 < \epsilon < G$. We then define

$$E_i := \{x : i\epsilon \leq g(x) < (i+1)\epsilon\} \cap K, \quad \text{for } 0 \leq i < G/\epsilon. \text{ The } E_i \text{ are disjoint.}$$

Each E_i, a Borel set, is in Σ. We can find open $U_i \supseteq E_i$ such that $\mu(U_i) < \mu(E_i) + \epsilon^2/G$, and find open $W_i \supseteq E_i$ such that $g(x) < (i+2)\epsilon$ in W_i. We then set $V_i := U_i \cap W_i$, so that the V_i satisfy both conditions. There are $h_i \leq V_i$ such that $\sum h_i = 1$ on K. Then

$$(*) \quad \ell(g) = \sum_{0 \leq i < G/\epsilon} (i+1)\epsilon \ell(h_i g((i+1)\epsilon)^{-1}) \leq \sum_{0 \leq i < G/\epsilon} (i+1)\epsilon\mu(V_i) < \sum_{0 \leq i < G/\epsilon} (i+1)\epsilon\mu(E_i) + \sum_{0 \leq i < G/\epsilon} (i+1)\epsilon^3/G.$$

Let $g_\epsilon := \sum_{0 \leq i < G/\epsilon} (i+1)\epsilon[x \in E_i]$; the first sum on the far right is $\int_X g_\epsilon d\mu$. The second sum there is less than $(G/\epsilon + 1)^2\epsilon^3/G < 4G\epsilon$.
By continuity g is measurable with respect to Σ. Since $g \leq g_\epsilon \leq g + \epsilon[x \in K]$, $\int_X g d\mu \leq \int_X g_\epsilon d\mu \leq \int_X g d\mu + \epsilon\mu(K)$.

By $(*)$,
$$\ell(g) \geq \sum_{0 \leq i < G/\epsilon} i\epsilon\mu(E_i) = \int_X g_\epsilon d\mu - \epsilon\mu(K) \geq \int_X g d\mu - \epsilon\mu(K).$$

Combining, we have
$$\int_X g d\mu - \epsilon\mu(K) \leq \ell(g) < \int_X g d\mu + \epsilon\mu(K) + 4G\epsilon.$$

Thus $\ell(g) = \int_X g\, d\mu$. Now given any function $F \in C_c(X)$ we can write

$$F = \operatorname{Re} F + i \operatorname{Im} F = (\operatorname{Re} F)^+ - (\operatorname{Re} F)^- + i((\operatorname{Im} F)^+ - (\operatorname{Im} F)^-) \text{ and apply the linearity of } \ell \text{ and } g \mapsto \int_X g\, d\mu.$$

Remark: No "continuity" assumption about ℓ was made. To prove results for non-positive linear functionals we need assumptions that allow us to express the linear functional in terms of positive ones, and some sort of continuity is used to deduce that. I wanted to use this form: $\ell : C_c(X) \to \mathbb{C}$ (linear!) is continuous "locally" if it is true that for every sequence $\{F_k\}$ in $C_c(X)$ whose supports are contained in a fixed K and tends uniformly to zero, $\ell(F_n) \to 0$. This is equivalent to requiring that for every K there is a constant C_K such that $|\ell(F)| \leq C_K \|F\|_\infty$ if $\operatorname{supp} F \subseteq K$. The texts I looked at did not contain this! I even looked in Dunford-Schwartz and did not find it (it *could* be there). The other forms would then follow from this one: bounded with respect to the maximum norm on $C_c(X)$, and bounded with respect to the maximum norm on $C_0(X)$ – functions tending to zero "at infinity."