Special Problem 1: Due Sept 20

Given $z = x + iy \neq 0$ use computation (algebra, the quadratic formua, etc.) to find all $w = u + iv$ such that $w^2 = z$. In this problem, an ambiguous sign $(\pm)$ that appears in one part of your answer must agree with all other ambiguous signs. We use $\mp$ to denote the sign opposite to $\pm$. Do not use the polar form!

$w^2 = z$ is the same as $x + iy = u^2 - v^2 + 2iuv$, so $x = u^2 - v^2$ and $y = 2uv$ must be true if $w^2 = z$. Let us therefore assume that a solution exists and continue with steps that are logically equivalent to $w^2 = z$.

If $y \neq 0$ then neither u nor v can be zero, so $v = y/2u$. Thus $x = u^2 - y^2/4u^2$ and so $4u^4 - 4xu^2 - y^2 = 0. =$ This quadratic in $u^2 > 0$ has the real root $u^2 = \frac{4x+\sqrt{16x^2-(-16y^2)}}{8} = \frac{x+\sqrt{x^2+y^2}}{2} = \frac{\sqrt{x^2+y^2}+x}{2}$ (the possible negative sign in the numerator cannot occur, since $x < \sqrt{x^2 + y^2}$ and we require that $u^2 > 0$). Then $v^2 = u^2 - x = \frac{\sqrt{x^2+y^2}-x}{2}$.

We return to the equation $y = 2uv$. If $y > 0$ then u and v must have the same sign, and their signs are opposite if $y < 0$. Thus $\operatorname{sgn}(v) = \operatorname{sgn}(y)\operatorname{sgn}(u)$. We can now write

$$u = \pm\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} \quad \text{and} \quad v = \pm\operatorname{sgn}(y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}, \quad \text{where the signs } \pm = \operatorname{sgn}(u) \text{ agree.}$$

We have obtained formulas for u and v in terms of x and y, provided that $y \neq 0$. However, our steps are reversible so it is indeed true that with the found values of u and v, $w^2 = z$. We have two solutions.

If $y = 0$ then $x \neq 0$ and at least one of u and v must be zero, but both cannot be zero. If $x > 0$ then $v = 0$ because otherwise $x < 0$ and if $x < 0$ then $u = 0$. We find then that $u = \pm\sqrt{x}$ and $v = 0$ if $x > 0$ and $u = 0$ and $v = \pm\sqrt{-x}$ if $x < 0$. Once again we have two solutions.

Are these the only solutions? Yes, because any complex number w such that $w^2 = z$ leads to being one of two solutions, as we just showed, the form of the solution depending on whether y is zero or not zero. We can (with care) include the case $y = 0$ in the $y > 0$ case, using continuity:

$$\text{with } y > 0, \quad u(x, y) = \sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} \quad \text{and} \quad v(x, y) = \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}},$$

the continuity of the square root function leads to

$$u(x, 0) = \sqrt{\frac{|x|+x}{2}} \quad \text{and} \quad v(x, 0) = \sqrt{\frac{|x|-x}{2}}, \quad \text{formulas that agree with what we did.}$$