Special Problem 2: *Prove that, if a and b are unequal natural numbers, then $|a-b| \geq 1$.*

We will use this inductive set: $S := \{1\} \cup \{x \in \mathbb{R} : x \geq 2\} (= \{1\} \cup [2, \infty)\,)$. To verify that S is inductive we note that $1 \in S$ by definition. Now suppose that $x \in S$. Then $x = 1$ or $x \geq 2$. If $x = 1$, then $x + 1 = 2 \in S$ by defintion. If $x \geq 2$, then $x + 1 > 2$, so $x + 1 \geq 2$ is true, hence $x + 1$ belongs to S, by definition. Hence S is an inductive set. This inductive set provides us with two useful deductions: (1) There is no natural number less than 1 and (2) there is no natural number n such that $1 < n < 2$.

We will also need to know that if $n > 1$ and n is a natural number, then $n - 1$ is a natural number. To show this, we will show that the set

$$S_1 := \{1\} \cup \{n \in \mathbb{N} : n - 1 \in \mathbb{N}\}$$

is inductive, so $S_1 = \mathbb{N}$. Then every natural number greater than one is in the set of those that can be reduced by one and remain natural numbers.

To verify that S_1 is inductive, we note that $1 \in S_1$ by definition. If $m \in S_1$, then $m = 1$ or $m - 1 \in \mathbb{N}$. If $m = 1$, then $n := m + 1 = 2$ satisfies $n - 1 = m = 1 \in \mathbb{N}$. That is, in this case, $m + 1 \in \mathbb{N}$. If instead $m \neq 1$, then $m - 1 \in \mathbb{N}$, so $(m + 1) - 1 = m \in \mathbb{N}$. Thus in either case, $m \in S_1 \Rightarrow m + 1 \in S_1$. Hence S_1, being an inductive set of natural numbers, consists of *all* natural numbers: $S_1 = \mathbb{N}$.

Now to solve our problem. We will use a "contradiction" approach. We assume that there exist natural numbers a and b such that $0 < |a - b| < 1$.

We define $E := \{e \in \mathbb{N} : (\exists f \in \mathbb{N})(e - 1 < f < e)\}$. Our set E will thus contain the larger of a and b. Hence E is non-empty.

We can now examine an *arbitrary* element $e_1 \in E$, and obtain $f_1 \in \mathbb{N}$ such that $e_1 - 1 < f_1 < e_1$. We cannot have $e_1 = 1$, since $f_1 \in \mathbb{N}$ and $f_1 < e_1$. Hence $e_1 > 1$, so by our discussion of the set S_1, we know that $e_1 - 1 \in \mathbb{N}$. It is also true that $e_1 - 1 \neq 1$, because then $1 = e_1 - 1 < f_1 < 2 = e_1$, so that $1 < f_1 < 2$, which is impossible, as shown by our deduction (2) about the set S that we defined. But then $e_1 - 1 > 1$. Thus $(e_1 - 1) - 1$ is a natural number. We also know that $e_1 - 1 < f_1$, so f_1 is a natural number that is greater than 1. Hence $f_1 - 1$ is a natural number.

Thus if we define $e_2 := e_1 - 1$ and $f_2 := f_1 - 1$, we see that $e_2 \in E$, providing f_2 as the "in between" natural number required for e_2 to qualify for membership in E.

We have shown: $(\forall e \in \mathbb{N})(e \in E \Rightarrow e - 1 \in E)$. Our set E is bounded below by zero (yes, it's bounded below by 1, but it won't matter) and is non-empty. Therefore the set $C := -E$ is non-empty and bounded above by 0, so there exists a least upper bound σ for C. We also know, about C, that $x \in C \Rightarrow x + 1 \in C$. Now we consider the number $\tau := \sigma - 1 < \sigma$, so there exists $c_1 \in C$ such that $\sigma - 1 < c_1$. That is, $\sigma < c_1 + 1 \in C$, which contradicts σ being an upper bound of C.

To escape the contradiction we must conclude that $E = -C$ is empty after all.

Corollary: $(\forall n \in \mathbb{N})(\forall k \in \mathbb{N})(k < n \Rightarrow k + 1 \leq n)$

Proof of the Corollary: For each natural number n let A_n denote the statement $(\forall k \in \mathbb{N})(k < n \Rightarrow k + 1 \leq n)$. A_1 is vacuously true. Suppose A_n is true. To show that A_{n+1} is true, we may assume $k < n + 1$, for otherwise the statement involving $k \geq n + 1$ is vacuously true. Among those $k < n + 1$, we can thus have $k < n$, or $k = n$, or $n < k < n + 1$. If $k < n$, A_n applies, and we may conclude that $k + 1 \leq n < n + 1$, so $k + 1 \leq n + 1$ is certainly true. If $k = n$ then $k + 1 = n + 1 \leq n + 1$. The last possibility, $n < k < n + 1$, cannot occur, by Special Problem 2 (especially when we look at that necessarily empty set E) .