Math 4606, Summer 2002 **Special Problem 2 – a solution**

Assignments are due at the start of class on the given Due date.

Please Note! *Special problems are like "term papers." They must be well-written, in ink, on standard 8.5 x 11 paper, and must be succint - with exactly enough detail.*

Paper torn from spiral notebooks is not acceptable.
Err in the direction of slightly excessive detail at first,
but prolixity is not acceptable.

Prove that the sequence defined by $x_0 = 1$ *and* $x_{n+1} = \sqrt{3 + 2x_n}$ *exists and converges. Find the limit.*

First, we notice that if $x > 0$ then $\sqrt{3 + 2x} > \sqrt{3} > 1 > 0$.

If we take $X := (0, \infty)$, $x^* = 1$ and $H(x.n) := \sqrt{3 + 2x}$ then since $H : X \to X$, by the Recursive Sequence theorem there exists a unique sequence $\{x_n\}$ such that $x_0 = x^* = 1$ and such that $x_{n+1} = H(x_n, n) = \sqrt{3 + 2x_n}$. Thus the sequence exists.

To prove the sequence converges we will show that it increases strictly and is bounded above.

We know that for all real numbers x and y, $x^2 - y^2 = (x - y)(x + y)$. Therefore, if x and y are both positive, so is $x + y$. Therefore $x^2 - y^2 = (x - y)(x + y) > 0$ if and only if $x - y > 0$.

We ask: is $x_{n+1} > x_n$? This will be true if and only if $x_{n+1}^2 > x_n^2$, meaning: is $x_{n+1}^2 - x_n^2 > 0$, or

$$\text{is } 3 + 2x_n > x_n^2? \text{ That is, is } x_n^2 - 2x_n - 3 < 0?$$

Since $x^2 - 2x - 3 = (x + 1)(x - 3)$, $x^2 - 2x - 3 < 0$ if and only if $x + 1$ and $x - 3$ have opposite signs. Since $x + 1 > x - 3$, this means $x + 1 > 0 > x - 3$, or $-1 < x < 3$. We have thus shown that

$$x_{n+1} > x_n \iff x_n^2 - 2x_n - 3 < 0 \iff -1 < x_n < 3.$$

Since we know that x_n is always positive, all we have to do to prove that $x_{n+1} > x_n$ for all n is to show that $x_n < 3$ for all n. This will then show that $\{x_n\}$ is increasing and bounded above.

To show that $x_n < 3$ for all n we use induction: $x_0 = 1 < 3$ and if $x_n < 3$ then $3 + 2x_n < 3 + 6 = 9$ so $x_{n+1} = \sqrt{3 + 2x_n} < \sqrt{9} = 3$. Hence $x_n < 3$ for all n, by induction.

Thus by the increasing-sequence -theorem we know that $\{x_n\}$ has a limit, that we call L: $\lim_{n\to\infty} x_n = L$. Moreover, $L \geq 1$ since $x_n > 1$ when $n > 1$.

We recall now that the square root function is continuous (when its argument is positive). Thus we have

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{3 + 2x_n} = \sqrt{3 + 2 \lim_{n\to\infty} x_n} = \sqrt{3 + 2L}.$$

Hence L is a positive root of $x^2 - 2x - 3 = (x + 1)(x - 3) = 0$, so $L = 3$.