Special Problem 3: Due Oct 10

Problem (39) in the Inductive Sets notes.

(39) **Problem:** *Prove that if $m \in \mathbb{N}$ and $n \in \mathbb{N}$ and $m \neq n$ then there exists no function $h : F_n \to F_m$ that is both one–to–one and onto.*

Contents

Introduction

This Introduction contains material that is for both proofs, the long one and the short one.

Those who feel the need to read the long solution are urged to at least skim the sections on Functions and the Lemma on modifying a function first.

If either of m and n is zero and $m \neq n$ there cannot exist a function $h : F_n \to F_m$ that is one–to–one and onto. This is covered in the Functions section. We therefore assume in what follows that both of m and n are positive.

We proceed by assuming the contrary. Thus we have $m \in \mathbb{N}$ and $n \in \mathbb{N}$ and $m \neq n$ and a function $h : F_n \to F_m$ that is both one–to–one and onto. There are two cases to consider: $n < m$ and $m < n$ (because $\mathbb{N} \subseteq \mathbb{R}$ so trichotomy holds). Let us reduce the cases to one. In case $m < n$ we interchange the *names* of these elements of $\mathbb{N}$ and replace h by h^{-1}, the function inverse to h (covered in the Functions section), and rename *that* function h.

Our "contrary assumption" allows us to assert that the set C of all positive integers n such that there exist an integer $m > n$ and a function $h : F_n \to F_m$ that is one–to–one and onto is non-empty. By the Well–Ordering Property of $\mathbb{N}$, C has a least element, that we also call n. We then obtain, by our assumption, some integer $m > n$ and a function $h : F_n \to F_m$ that is one–to–one and onto. Moreover, n is the smallest positive integer with this "contrary" property.

Let us observe that $n > 1$ is necessary (so that $n - 1 \geq 1$). Otherwise, every ordered pair in the function h has first component 0 since $F_1 = \{0\}$. But then, since $0 \neq 1$, the ordered pairs $(0, 0)$ and $(0, 1)$ could not be in h, for this would contradict a property that a function must have (the "vertical line test"). Since $m > 1$, F_m contains both 0 and 1, so both of these numbers must be second components of an ordered pair in h, for h is assumed to be onto.

Functions

Given sets X and Y we denote by $X \times Y$ the set of *all ordered pairs* (x, y), with $x \in X$ and $y \in Y$. As an Exercise, imagine or write down $EG := \{1, 2\} \times \{1, 2, 3\}$. A *relation* is a subset of $X \times Y$. In particular, $\emptyset$ and $X \times Y$ are relations. A relation $R \subseteq X \times X$ is called a *relation on* X. A very important relation on X is the *diagonal,* or the *identity (relation)* defined by $\Delta := \{(x, x) : x \in X\}$. A *function* f *from* X *into* Y is a special kind of relation. Thus f is a set of ordered pairs.

Definition: *A function from X into Y is a subset f of $X \times Y$ with the following properties:*
(1) For all $x \in X$ there exists $y \in Y$ such that $(x, y) \in f$.
(2) For all $x \in X$, for all $y_1 \in Y$ and for all $y_2 \in Y$, if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$.

Property (2) is called "the vertical–line test." Property (1) might be called "the Rule assigning to each $x \in X$ a $y \in Y$." However, (1) did not specify that only one $y \in Y$ is assigned to x. That is what (2) does. As an Exercise, write down non-empty relations $R_1 \subseteq EG$ and $R_2 \subseteq EG$ such that R_1 has Property (1) and R_2 has Property (2) and neither one is a function. Also, verify that the identity relation is a function from X into X, also known as *the identity function.*

Notation for functions, and related definitions

We do not usually think of functions as sets of ordered pairs. But we do think of the graphs of functions, especially real–valued functions of one real variable. That "pictorial" view includes our ordered–pair definition because we usually think of a point in the plane as having an x coordinate, or component, and a y coordinate, or component. On the sketch of a graph, these points are usually identified with a label "$y = f(x)$" or $(x, f(x))$ or even (x, y). We often call the first, or x component of $(x, y) \in f$ an "argument," or even an "input" and we call the second componenet y a "value," or even an "output" of the function f. When x is left as a variable, we also write "the function $f(x)$." If f is a function from X to Y we call X the *domain* (space) of f and we call Y the *range* (space) of f. The latter term is not universal yet. Some people use "the range of f" to mean the set of all $y \in Y$ such that $y = f(x)$ for at least one $x \in X$. We will use the term "image of f" or "the image of X under f," or "with respect to f" and so on. There is more in **2.2,** in Rudin's book. It is also very convenient to write "$f : X \to Y$" and read it as (function) f mapping X into Y," along with other convenient ways of saying the same thing. We will denote the identity function by $I : X \to X$, where $I(x) := x$ for all $x \in X$. When another set is under discussion as well we write I_X for this function. Sometimes i is used to denote the identity function.

We will define "onto function" and "one–to–one function" now. The "official" definitions, in terms of ordered pairs, will be followed, informally, by the ones you may be familiar with already.

Definition: *A function $f : X \to Y$ is onto or is surjective if for all $y \in Y$ there exists $x \in X$ such that $(x, y) \in f$.*

Definition: *A function $f : X \to Y$ is one–to–one or is injective if for all $x_1 \in X$, for all $x_2 \in X$ and for all $y \in Y$, if $(x_1, y) \in f$ and $(x_2, y) \in f$, then $x_1 = x_2$.*

Informally, f is onto if every y in Y is $f(x)$ for some x in X; f is one–to–one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. The latter is equivalent to saying that different inputs to f result in different outputs.

Let us notice now that if X is empty and Y is not then f cannot be onto. Let us study in some detail (because it is relevant to our problem!) what happens when one (or more) of X and Y is empty.

If either one of X or Y is empty, then $X \times Y = \emptyset$ because there is no element of a set to serve as a *first component* of an ordered pair, if $X = \emptyset$ and no element of a set to serve as a *second component* of an ordered pair, if $Y = \emptyset$. If $X = \emptyset$ then Properties (1) and (2) hold vacuously so $f = \emptyset$ is a function from $X = \emptyset$ to Y, no matter what set Y may be, including $Y = \emptyset$. In particular, the set of all functions from $X = \emptyset$ to Y, where Y is any set whatsoever, is $\{\emptyset\}$. But if X is not empty and Y is empty, Property (1) cannot hold, so no function from $X \neq \emptyset$ to $Y = \emptyset$ exists. In particular, the set of all functions from $X \neq \emptyset$ to $Y = \emptyset$ is $\emptyset$.

The application to our problem is that, if one, but not both, of m and n is zero, then there is either *no* function $h : F_n \to F_m$ or no function $h : F_n \to F_m$ that is *onto.*

Next, we discuss *composite functions, one–to–one correspondences, inverse relations* and *inverse functions.* Rudin's **2.2** also discusses *inverse images.*

The composition of two functions may not exist; they have to be related appropriately. The inverse (function) of a function also may not exist. But the inverse of a relation always exists.

Definition: *If $R \subseteq X \times Y$ then $R^{-1} \subseteq Y \times X$ is defined by*

$$R^{-1} := \{(y, x) \in Y \times X : (x, y) \in R\}.$$

All we do is reverse the first and second components of the pairs in R.

Definition: *If $f : X \to Y$ and $g : Y \to Z$, then the composite function $g \circ f : X \to Z$ is defined by*

$$g \circ f := \{(x, z) \in X \times Z : \ \textit{there exists} \ \ y \in Y \ \textit{such that} \ \ (x, y) \in f \ \textit{and} \ (y, z) \in g\}.$$

Since this "Definition" depends on the truth of statements "outside" the definition itself, we have to verify that it makes sense! Also, note that g and f are "turned around."

Proposition: *$g \circ f$ is a function from X into Z.*

Proof: We have to verify Properties (1) and (2) of a function. Given $x \in X$, Property (1) for f is true, so there exists $y \in Y$ such that $(x, y) \in f$. Moreover, y is unique, by Property (2). Since $y \in Y$, Property (1) holds for g so there exists a unique $z \in Z$ (Property (2) for g) such that $(y, z) \in g$. Thus Property (1) holds for $g \circ f$. Property (2) holds as well, but it's not shown directly. Just to show we can, let's verify directly that Property (2) holds for $g \circ f$. Given $x \in X$, suppose that $(x, z_1) \in g \circ f$ and $(x, z_2) \in g \circ f$. Then there exist $y_1 \in Y$ and $y_2 \in Y$ such that $(x, y_i) \in f$ and $(y_i, z_i) \in g$, for $i = 1$ and $i = 2$. By Property (2) for f, $y_1 = y_2$. Let us define $y := y_1 = y_2$. Then $(y, z_i) \in g$, for $i = 1$ and $i = 2$. By Property (2) for g, $z_1 = z_2$, so Property (2) holds for $g \circ f$. Thus $g \circ f$ is a function from X into Z.

Definition: *A function $f : X \to Y$ is a one–to–one correspondence, or a bijection, if f is both one–to–one and onto.* The name suggests that the elements of X and Y can be matched in a unique way.

Definition: *A function $f : X \to Y$ has an inverse(function) if there exists $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. We write $g = f^{-1}$ and may write $f = g^{-1}$ as well.*

Theorem: *A function $f : X \to Y$ has an inverse if and only if f is a one–to–one correspondence.*

Proof: Suppose first that $f : X \to Y$ has an inverse. Then there exists $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. We will use "$y = f(x)$" (and so on) instead of ordered pairs. To show that f is one–to–one we suppose that $f(x_1) = f(x_2)$, where $x_1 \in X$ and $x_2 \in X$. Since $g \circ f = I_X$, $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$ so f is one–to–one. To show that f is onto we let $y \in Y$ be given. Since $f \circ g = I_Y$, $f(g(y)) = y$, so f is onto.

Next we suppose that f is a one–to–one correspondence. We now use the inverse relation $f^{-1} \subseteq Y \times X$ (we recall that it's $\{(y, x) \in Y \times X : (x, y) \in X\}$). We need to show that $g := f^{-1}$ is a function and that g satisfies the conditions in the Theorem's statement. The proof is very straightforward, provided that we realize that "onto" for f, when applied to the ordered pairs in f^{-1} is precisely Property (1) for g, and that "one–to–one," applied to the ordered pairs in f^{-1} is precisely Property (2) for g. Here are the details, though it would be better for you to do them yourself. Given $y \in Y$ we know that there exists $x \in X$ such that $f(x) = y$, since f is onto. Therefore $(x, y) \in f$, so $(y, x) \in f^{-1} = g$, so Property (1) holds for g. Suppose that $(y, x_1) \in g$ and $(y, x_2) \in g$. Then $(x_1, y) \in f$ and $(x_2, y) \in f$. That is, $f(x_1) = f(x_2)$. Since f is one–to–one, $x_1 = x_2$. This shows that Property (2) holds for g, so that g is a function from Y into X.

It remains to show that $g \circ f = I_X$ and $f \circ g = I_Y$. By definition

$$g \circ f := \{(x, z) \in X \times X : \text{ there exists } \ y \in Y \ \text{ such that } \ (x, y) \in f \ \text{ and } \ (y, z) \in g\}.$$

But $(y, z) \in g$ means $(z, y) \in f$. Since $(x, y) \in f$ and f is one–to–one, $z = x$. Thus $g \circ f(x) = x$.
Again by definition

$$f \circ g := \{(y, z) \in Y \times Y : \text{ there exists } \ x \in X \ \text{ such that } \ (y, x) \in g \ \text{ and } \ (x, z) \in f\}.$$

For all $y \in Y$ there exists $x \in X$ such that $f(x) = y$, so $(x, y) \in f$ and thus $(y, x) \in g$. Hence the diagonal, Δ, is contained in $f \circ g$. Property (2) for f implies that if, also, $(x, z) \in f$ then $y = z$. Thus the only points $(y, z) \in f \circ g$ lie on the diagonal, and thus $f \circ g = \Delta = I_Y$. This completes the proof.

A Lemma on modifying a function

Lemma: *Let $h : X \to Y$ and suppose that there exist $x_1 \in X$ and $x_2 \in X$ such that $x_1 \neq x_2$. Then there exists a unique function $h_1 : X \to Y$ such that for all $x \in X$, $h_1(x) = h(x)$ if $x \neq x_1$ and $x \neq x_2$, and such that $h_1(x_1) = h(x_2)$ and $h_1(x_2) = h(x_1)$. If h is one–to–one, so is h_1. If h is onto, so is h_1.*
Proof: A perfectly general "ordered–pairs" version of the proof can be found in "A lengthy solution with many details," under the heading "Here we return to the modification of the function h." We give a different, trivial, version, with exactly the same idea, as long as you accept, as the definition of equality for functions h and h_1 the equality of their domains and ranges, and that for every x in the domain, $h(x) = h_1(x)$. This is the definition we will use. It may be controversial.

We now define h_1 by specifying its values, at each point $x \in X$, as in the statement of the Lemma. We can do this unambiguously since $x_1 \neq x_2$. Since we have specified a unique value for $h_1(x)$, for every $x \in X$, our definition defines a function h_1. If some othe function, h_2, satisfied all these conditions, it would also satisfy the conditions for being equal to h_1.

Suppose h is one–to–one. Then $h_1(x') = h_1(x'')$ certainly implies that $x' = x''$ if neither point is in $\{x_1, x_2\}$, since h is one–to–one. The same argument applies if both points are in $\{x_1, x_2\}$. If one of the points is in $\{x_1, x_2\}$ and the other is not, they are not equal. The values under h are thus different at x' and at x''. If $x'' \in \{x_1, x_2\}$ the possible values for $h_1(x'')$ are $h(x_1)$ and $h(x_2)$. Since neither of these is the value of $h(x') = h_1(x')$, we see that $h_1(x') \neq h(x'')$. This is the contrapositive of our definition of "one–to–one," and this completes the proof of the one–to–one case. Suppose now that h is onto. Let $y \in Y$. Then there exists $x \in X$ such that $h(x) = y$. If $x \notin \{x_1, x_2\}$ then $h_1(x) = h(x) = y$. If $x \in \{x_1, x_2\}$ then $y = h_1(x_2)$ if $x = x_1$, and $y = h_1(x_1)$ if $x = x_2$, by the way h_1 was defined. Thus h_1 is onto if h is onto. This completes the proof of the Lemma.

A lengthy solution with many details

Reminder: this part *continues* the Introduction.

We will modify our function h in such a way that the *modified* h, call it h_1, has all the properties of h, and in addition the property that $h_1(0) = 0$. This will allow us to carry out some steps that lead to a function $h_2 : F_{n-1} \to F_{m-1}$ that is one–to–one and onto. This will give us the desired contradiction: C contains an element smaller than its least element.

This part covers the "steps to carry out," and can be skipped at first reading

In fact, let us consider the possiblity that $h(0) = 0$. In fact, this cannot be true, for then we could remove the ordered pair $(0, 0)$ from h. Let us show that if we replace the *remaining* ordered pairs (x, y) in h by the ordered pairs $(x-1, y-1)$ we obtain a function $h_2 : F_{n-1} \to F_{m-1}$ that is one–to–one and onto. First we note that since $(0, 0) \in h$ and h is one–to–one, no *other* ordered pair (x, y) in h has 0 as either component. For $(x, y) \neq (0, 0)$ means that $x \neq 0$ or $y \neq 0$. If $x \neq 0$, then $y = h(x) \neq h(0) = 0$. If $y \neq 0$, then $x \neq 0$ because $h(0) = 0 \neq y$. Thus in this function h, $(x, y) \neq (0, 0)$ means $x \neq 0$ *and* $y \neq 0$. Since each of x and y is therefore a positive integer, $x-1$ and $y-1$ are natural numbers. Moreover, $0 \leq x-1 \leq (n-1)-1$ and $0 \leq y-1 \leq (m-1)-1$. We now officially define

$$h_2 := \{(u, v) \in F_{n-1} \times F_{m-1} : (u, v) = (x-1, y-1), \text{ where } (x, y) \in h \text{ and } (x, y) \neq (0, 0)\}.$$

To show that h_2 is a function suppose $u \in F_{n-1}$. Then with $x := u+1$ there exists (x, y) in h and $x \neq 0$ such that $(u, v) = (x-1, y-1)$, by the definition of h_2. If $(u, v_1) \in h_2$ and $(u, v_2) \in h_2$ then $(u+1, v_1+1) \in h$ and $(u+1, v_2+1) \in h$ so $v_1 = v_2$. Thus h_2 is a function with domain F_{n-1} and range space F_{m-1}. To see that h_2 is one–to–one and onto, suppose $u_1 \neq u_2$ are in F_{n-1}. Then $h_2(u_1) = h(u_1+1) \neq h(u_2+1) = h_2(u_2)$ because h is one–to–one. Next suppose that $v \in F_{m-1}$. Then $y := v+1 \in F_m$ so there exists $x \in F_n$ such that $y = h(x) = h_2(x-1)+1$, so $y-1 = v = h_2(x-1) = h_2(u)$, where $u := x-1 \in F_{n-1}$. This gives our contradiction, and shows that, under our contrary assumption, $h(0) \neq 0$.

Here we return to the modification of the function h

As shown in the part you may have skipped, under our contrary assumption, $h(0) \neq 0$. We will modify the function h by changing two of its ordered pairs. The modified function, h_1, will also be a one–to–one and onto mapping from F_n onto F_m, and it will be true that $h_1(0) = 0$. This will give us the contradiction we want.

According to intuition the contradiction is going to come about because h is onto. Since $n < m$, $n \in F_m$ but $n \notin F_n$. Therefore the set $E := \{k \in \mathbb{N} : k \in F_n \text{ and } h(k) \geq n\}$ is nonempty. Let us define $D \subseteq F_n$ by $D = F_n \setminus E$. Then $D \neq \emptyset$ because $h(\ell) = 0$ for some $\ell \in F_n$ because h is onto, and then $\ell \in D$. We note that $h(D) = F_n$ and $h(E) = F_m \setminus F_n$.

There are two cases to consider. In Case 0, $0 \in D$, and in Case 1, $0 \in E$. Since D and E have no points in common and since $D \cup E = F_n$, these two cases are exhaustive.

Case 0 Since $0 \in D$ but $h(0) \neq 0$ [so that $(0, h(0)) \neq (0, 0)$] there is $d_0 \in D$ such that $d_0 \neq 0$ and $(d_0, 0) \in h$ [i.e., $h(d_0) = 0$]. We now define a relation $h_1 \subseteq F_n \times F_m$ by removing the ordered pairs $(0, h(0))$ and $(d_0, 0)$ from h and replacing them by $(0, 0)$ and $(d_0, h(0))$. In symbols,

$$h_1 := [h \setminus \{(0, h(0)), (d_0, 0)\}] \cup \{(0, 0), (d_0, h(0))\}.$$

We notice this: if $x \notin \{0, d_0\}$, then $(x, h(x)) \in h_1$. Moreover, $\{(0, 0), (d_0, h(0))\} \subseteq h_1$.

We can show that h_1 is a function (to be done soon). But then since $(0, 0) \in h_1$ we have $h_1(0) = 0$. We will also show that h_1 is a one–to–one and onto mapping from F_n onto F_m, which will lead to a contradiction, as shown in the "steps to carry out" part you may have skipped. Once we have shown that h_1 is a one–to–one and onto mapping from F_n onto F_m, it will follow from our contrary assumption that Case 0 cannot hold.

Now we show that h_1 is a one–to–one and onto mapping from F_n onto F_m. This will be done carefully, using the definition of function as a set of ordered pairs.

(1) We need to show that: for every $x \in F_n$ there exists $y \in F_m$ such that $(x, y) \in h_1$.
In case $x \neq 0$ and $x \neq d_0$, the ordered pair $(x, h(x)) \in h_1$, because $(x, h(x)) \in h$ and the only ordered pairs that were changed had first component 0 or d_0. If $x = 0$ or $x = d_0$, the ordered pairs $(0, 0)$ and $(d_0, h(0))$ satisfy the condition in (1).
(2) We need to show that: for every $x \in F_n$, every $y_1 \in F_m$ and every $y_2 \in F_m$, if $(x, y_1) \in h_1$ and $(x, y_2) \in h_1$ then $y_1 = y_2$.
If $x \neq 0$ and $x \neq d_0$ and the ordered pairs (x, y_1) and (x, y_2) are in h_1 they are also in h. Since h is known to be a function, $y_1 = y_2$. If $x \in \{0, d_0\}$, then $y_1 = y_2$ by the way h_1 was defined (since h is one–to–one).

Thus h_1 is a function from F_n into F_n. Thus if $x \notin \{0, d_0\}$, then $h_1(x) = h(x)$. Moreover, $h_1(0) = 0$ and $h_1(d_0) = h(0)$. Next, we need to show that h_1 is one–to–one and onto. Once again we use the ordered–pairs definition of *function,* but not as strongly. We show first that h_1 is one–to–one, then that h_1 is onto.

(3) We need to show that: for every $x_1 \in F_n$, every $x_2 \in F_n$ and every $y \in F_m$, if $(x_1, y) \in h_1$ and $(x_2, y) \in h_1$ then $x_1 = x_2$. In other words, we need to show that $h_1(x_1) = h_1(x_2)$ implies $x_1 = x_2$.
If both of x_1 and x_2 are in $\{0, d_0\}$, then, because $h(0) \neq 0$ we must have $x_1 = x_2$, since $h(0)$ and 0 include the second members that the ordered pairs $(x_1, y) \in h_1$ and $(x_2, y) \in h_1$ have. If neither of x_1 and x_2 is in $\{0, d_0\}$, the ordered pairs (x_1, y) and (x_2, y) are in h as well as h_1. Thus $x_1 = x_2$ because h is one–to–one. Finally, if one of x_1 and x_2 is in $\{0, d_0\}$ and the other is not, then $x_1 \neq x_2$. Since h is one–to–one, if $x \notin \{0, d_0\}$ then $h(x) \neq 0 = h(d_0)$ and $h(x) \neq h(0)$. Thus $h_1(x_1) \neq h_1(x_2)$ and we have shown the contrapositive of the condition in (3) is true. Thus h_1 is one–to–one.
(4) We need to show that: for every $y \in F_m$ there exists $x \in F_n$ such that $(x, y) \in h_1$. That is, we need to show that for every $y \in F_m$, the equation $y = f(x)$ has a solution in F_m.
Let $y \in F_m$ be given. Then, because h is onto, there exists $x \in F_n$ such that $h(x) = y$. If $x \notin \{0, d_0\}$, then $h_1(x) = h(x) = y$. If $x \in \{0, d_0\}$ and $x = 0$, then $y = h(x) = h(0) = h_1(d_0)$. Otherwise, $x = d_0$ and so $y = h(d_0) = 0 = h_1(0)$. Thus h_1 is onto.

Under our contrary assumption, this cannot be true. Hence it follows from our contrary assumption that Case 0 does not hold, that is, it must be true that $0 \in E$, the k's in F_n such that $h(k) \notin F_n$.

We turn finally to Case 1: $0 \in E$. This means that $h(0) \geq n$. Since h is onto, there exists $d_1 \in D$ such that $h(d_1) = 0$. Now we define a relation $h_1 \subseteq F_n \times F_m$ by

$$h_1 := [h \setminus \{(0, h(0)), (d_1, 0)\}] \cup \{(0, 0), (d_1, h(0))\}.$$

When in Case 0 we proved that the h_1 defined there was a function, we used an argument that we now adapt to the present h_1 by changing all the terms d_0 to d_1. In our present case, the modification removes 0 from E and replaces it by d_1; it removes d_1 from D and replaces it by 0. Here is the modified argument:

(1) We need to show that: for every $x \in F_n$ there exists $y \in F_m$ such that $(x, y) \in h_1$.
In case $x \neq 0$ and $x \neq d_1$, the ordered pair $(x, h(x)) \in h_1$, because $(x, h(x)) \in h$ and the only ordered pairs that were changed had first component 0 or d_1. If $x = 0$ or $x = d_1$, the ordered pairs $(0, 0)$ and $(d_1, h(0))$ satisfy the condition in (1).
(2) We need to show that: for every $x \in F_n$, every $y_1 \in F_m$ and every $y_2 \in F_m$, if $(x, y_1) \in h_1$ and $(x, y_2) \in h_1$ then $y_1 = y_2$.
If $x \neq 0$ and $x \neq d_1$ and the ordered pairs (x, y_1) and (x, y_2) are in h_1 then they are also in h. Since h is known to be a function, $y_1 = y_2$.

Thus h_1 is a function from F_n into F_n. Thus if $x \notin \{0, d_1\}$, then $h_1(x) = h(x)$. Moreover, $h_1(0) = 0$ and $h_1(d_1) = h(0)$. Next, we need to show that h_1 is one–to–one and onto. Once again we use the ordered–pairs definition of *function,* but not as strongly. We show first that h_1 is one–to–one, then that h_1 is onto.

(3) We need to show that: for every $x_1 \in F_n$, *every* $x_2 \in F_n$ *and every* $y \in F_m$, *if* $(x_1, y) \in h_1$ *and* $(x_2, y) \in h_1$ *then* $x_1 = x_2$. In other words, we need to show that $h_1(x_1) = h_1(x_2)$ implies $x_1 = x_2$.
If both of x_1 and x_2 are in $\{0, d_1\}$, then, because $h(0) \neq 0$ we must have $x_1 = x_2$, since $h(0)$ and 0 include the second members that the ordered pairs $(x_1, y) \in h_1$ and $(x_2, y) \in h_1$ have. If neither of x_1 and x_2 is in $\{0, d_1\}$, the ordered pairs (x_1, y) and (x_2, y) are in h as well as h_1. Thus $x_1 = x_2$ because h is one–to–one. Finally, if one of x_1 and x_2 is in $\{0, d_1\}$ and the other is not, then $x_1 \neq x_2$. Since h is one–to–one, if $x \notin \{0, d_1\}$ then $h(x) \neq 0 = h(d_1)$ and $h(x) \neq h(0)$. Thus $h_1(x_1) \neq h_1(x_2)$ and we have shown the contrapositive of the condition in (3) is true. Thus h_1 is one–to–one.
(4) We need to show that: for every $y \in F_m$ *there exists* $x \in F_n$ *such that* $(x, y) \in h_1$. That is, we need to show that for every $y \in F_m$, the equation $y = f(x)$ has a solution in F_m.
Let $y \in F_m$ be given. Then, because h is onto, there exists $x \in F_n$ such that $h(x) = y$. If $x \notin \{0, d_1\}$, then $h_1(x) = h(x) = y$. If $x \in \{0, d_1\}$ and $x = 0$, then $y = h(x) = h(0) = h_1(d_1)$. Otherwise, $x = d_1$ and so $y = h(d_1) = 0 = h_1(0)$. Thus h_1 is onto.

We have nearly reached the contradiction we wanted, because $h_1(0) = 0$. The steps we carried out now apply to this h_1, so there exists a function $h_2 : F_{n-1} \to F_{m-1}$ that is one–to–one and onto. This contradicts that n is the least element of C. This completes the lengthy proof.

A much shorter solution

This part continues the Introduction.

Since $n \notin F_n$ but $n \in F_m$, the set $E := \{k \in F_n : h(k) \geq n\}$ is nonempty and is not all of F_n. There is, in E, some $K \in F_n$ such that $h(K) = m - 1$. There are two cases to consider: $K = n - 1$ and $0 \leq K < n - 1$.

If $K = n - 1$, we consider the set $G := h \setminus \{(n-1, m-1)\}$. For all $(k, \ell) \in G \subseteq F_n \times F_m$, $k < n - 1$ and $\ell < m - 1$. Thus $G \subseteq F_{n-1} \times F_{m-1}$, and we recall that, under our contrary assumption, $n - 1 > 0$. We now claim that G is a function from F_{n-1} into F_{m-1} that is one–to–one and onto. Once the claim is justified, we have found an element of C smaller than its least element, which gives a contradiction.

To justify the claim, we see first that, for all $k \in F_{n-1}$, the ordered pair $(k, h(k))$ is in G (it is in h and is not equal to $(n-1, m-1)$). Thus Property (1) holds (see the first Definition in the section on Functions). If (k, ℓ) is in $G \subseteq h$, then $k < n - 1$ and so $(k, \ell) = (k, h(k))$. Thus Property (2) holds. Thus G is a function from F_{n-1} into F_{m-1}.

Second, since G is the restriction of h to F_{n-1}, G is one–to–one.

Finally, to see that G is onto, the fact that h is onto allows us to say that, if $\ell < m - 1$, there exists $k \in F_n$ such that $h(k) = \ell$. Since h is one–to–one, $k \neq n - 1$, so $k \in F_{n-1}$. Thus G is onto. This gives the contradiction, so that this case (under our contrary assumption) cannot occur.

We are thus in the case $0 \leq K < n - 1$. Now we consider the ordered pairs $(K, m-1)$ and $(n-1, h(n-1))$, both in h. In our case, $x_1 := K \neq n - 1 =: x_2$. We now apply the Lemma on modifying a function (page 3), and obtain a function $h_1 : F_n \to F_m$ that is also one–to–one and onto, but now satisfies $h_1(n-1) = h_1(x_2) = h(x_1) = m - 1$. The function h_1 satisfies all the conditions that h did, but this one satisfies the case we studued first, that led to a contradiction. There are no more cases to consider, so we have our contadiction, and this completes the solution of Special Problem 3.