Special Problem 3: Due Oct 22

If $\{c_n\}$ is a sequence of complex numbers we define R, $0 \le R \le +\infty$, by

$$\frac{1}{R} := \limsup_{n\to\infty} |c_n|^{1/n} \text{ and we call } R \text{ the radius of convergence of the power series } \sum_{n=0}^{\infty} c_n z^n.$$

Verify that if $|z| < R$ the power series converges absolutely and that if $|z| > R$ the power series diverges.

If $R = 0$ then $\limsup |c_n|^{1/n} = +\infty$. Then for any $z \neq 0$ we have $\limsup |c_n z^n|^{1/n} = +\infty$ and thus $\limsup |c_n z^n| = +\infty$ as well, so the terms of the power series do not tend to zero if $z \neq 0$. The conclusion holds in this case.

If $R = +\infty$ we have $\limsup |c_n|^{1/n} = 0$ so for any $z \in \mathbb{C}$ we have $\limsup |c_n (2z)^n| = 0$ as well. This means that $\lim |c_n (2z)^n| = 0$ so eventually $|c_n (2z)^n| < 1$ and thus eventually $|c_n z^n| < 1/2^n$. The series converges by comparison with a geometric series and the conclusion holds in this case because there are no z with $|z| > R$.

If $0 < R < +\infty$ and we let $|z| < R$ then we observe that $\limsup |c_n|^{1/n} R = 1$. We conclude that $\limsup |c_n|^{1/n} |z| = |z|/R < 1$. Hence there is some r such that $\limsup |c_n|^{1/n} |z| = |z|/R < r < 1$. Thus eventually $|c_n|^{1/n} |z| < r$ so that $|c_n z^n| < r^n$ and the series converges absolutely by comparison to a geometric series. On the other hand, if $|z| > R$, $\limsup |c_n|^{1/n} |z| = |z|/R > 1$. Hence there is some r such that $\limsup |c_n|^{1/n} |z| = |z|/R > r > 1$. Thus for every N there is $n \ge N$ such that $|c_n|^{1/n} |z| > r$ so that $|c_n z^n| > r^n$ and thus there is a subsequence $\{n_k\}$ of values of n such that $|c_{n_k} z^{n_k}| \to +\infty$ and the series diverges rather badly!

Note: This solution made free use of the note *On* lim sup *and radius of convergence* that is linked to the class Web page. Please see me if you want more details.