Special Problem 4: Due Nov 5 (see **3.16**)

Prove that, if $\{s_n\}$ is a sequence of real numbers, then $\limsup_{n\to\infty} s_n = \lim_{n\to\infty} \sup_{m\geq n} s_m$ (this can be an *extended* real).

We know (by **3.17a**) there is a subsequence $\{s_{n_i}\}$ that has limit $s^* := \limsup_{n\to\infty} s_n$. Since $\sup_{m\geq n_i} s_m \geq s_{n_i}$ and $a_n := \sup_{m\geq n} s_m$ is a decreasing sequence, $\lim_{n\to\infty} \sup_{m\geq n} s_m = \lim_{i\to\infty} \sup_{m\geq n_i} s_m \geq \lim_{n\to\infty} s_{n_i} = s^*$. Thus $\limsup_{n\to\infty} s_n \leq \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\geq n} s_m$.

To show the other inequality, we consider two cases: $s^* = +\infty$, $s^* < +\infty$. In the first case there is a subsequence $\{s_{n_i}\}$ that has limit $+\infty$. Thus for all n, $\sup_{m\geq n} s_m = +\infty$ so we have equality immediately in this case. In the second case, there exist real numbers $x > s^*$, and for any such number x, there exists an integer N so large that for all integers $m \geq N$, $s_m < x$. Thus if $n \geq N$, $\sup_{m\geq n} s_m \leq x$. Hence $\lim_{n\to\infty} \sup_{m\geq n} s_m \leq x$. This is true for every real $x > s^*$. Thus $\lim_{n\to\infty} \sup_{m\geq n} s_m$ is a lower bound for the non-empty set (s^*, ∞). Hence $\lim_{n\to\infty} \sup_{m\geq n} s_m \leq s^*$.