Special Problem 4: Due Nov 8

Show that if $f(z)$ *is analytic in a domain* D *that contains the closed disc with center* z_o, *radius* R *and boundary* C *traced counterclockwise then for every* z *in the open disc in question and for every* $n \in \mathbb{N}$,

(1) $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Proof: We know that the Cauchy Integral Formula gives

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad \text{so if } \ 0 < |h| < \tfrac{1}{2}(R - |z - z_o|), \quad \text{then}$$

(2) $$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i h} \int_C \left(\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z} \right) d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)(\zeta - z - h)}\, d\zeta,$$

for the h in the denominator cancels in passing from the second term to the third. We notice that

(3) $$\frac{1}{(\zeta - z)(\zeta - z - h)} - \frac{1}{(\zeta - z)(\zeta - z)} = \frac{h}{(\zeta - z)^2(\zeta - z - h)} \quad \text{and we recall that for every } \ \zeta \in C$$

it is true that $|\zeta - z| \geq R - |z - z_o| > 0$ and that $|\zeta - z - h| \geq |\zeta - z| - |h| \geq \frac{1}{2}(R - |z - z_o|) > 0$. Hence

(4) $$\left| \frac{1}{(\zeta - z)(\zeta - z - h)} - \frac{1}{(\zeta - z)(\zeta - z)} \right| \leq \frac{|h|}{|\zeta - z|^2 |\zeta - z - h|} \leq \frac{2|h|}{(R - |z - z_o|)^3}.$$

Now when we use the triangle inequality for contour integrals,

(5) $$\left| \frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \leq \frac{1}{2\pi} \int_C \frac{2|h||f(\zeta)|}{(R - |z - z_o|)^3}\, |d\zeta| \leq \frac{2MR|h|}{(R - |z - z_o|)^3}, \quad \text{where}$$

M is an upper bound for $|f(\zeta)|$ on C, finite because f is continuous on the compact set C. Thus as $h \to 0$

$$\frac{f(z+h) - f(z)}{h} \to \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = f'(z), \quad \text{by the definition of complex derivative.}$$

Remarks: The assumption $|z + h - z_o| < \frac{1}{2}(R - |z - z_o|)$ was made so that we could get rid of the h in $|\zeta - z - h|$ in (4).

The line between (3) and (4) makes use of this tricky triangle-inequality variant:

$|z_1 - z_2| \geq |z_1| - |z_2|$, true because we can write $|z_1| = |(z_1 - z_2) + z_2|$, use the triangle inequality and

move $|z_2|$ to the other side of the inequality. Here, we had $z_1 = \zeta - z$ and $z_2 = h$.

Several steps were skipped in (5)! The difference quotient should have been replaced by the last integral in (2), then the integrals combined, (3) used, then the inequality in (4) should have been used in connection with the triangle inequality for contour integrals.

This part of the proof could have been shortened by *citing* uniform convergence due to uniform continuity. The steps done here actually make uniform convergence explicit by using the uniform inequality (4).

More remarkable still is the fact that this part could have been skipped entirely because the Cauchy Integral Formula is already the initial statement in an Induction proof.

To proceed by induction (we have proved the "Basis Case," $n = 1$) we assume that (1) holds for some n and for *all* analytic functions. We have shown (by having done the $n = 1$ case in class) that if a function is analytic then so are its derivatives, of all orders. Therefore if (1) holds for all analytic functions for some n then

$$f^{(n+1)}(z) = (f')^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f'(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \quad \text{for all} \quad z \quad \text{with} \quad |z - z_o| < R. \tag{6}$$

Let us integrate by parts on the right-hand side. First, we convert to the parametric form of the contour integral. To do this we write $\zeta(t) = z_o + Re^{i\theta}$, with $0 \le \theta \le 2\pi$. Then $d\zeta = iRe^{i\theta}d\theta = \zeta'(\theta)\,d\theta$ so we have

$$\int_C \frac{f'(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta = \int_0^{2\pi} \frac{f'(\zeta(\theta))\zeta'(\theta)}{(\zeta(\theta) - z)^{n+1}}\,d\theta = -\int_0^{2\pi} \frac{-(n+1)\zeta'(\theta)f(\zeta(\theta))}{(\zeta(\theta) - z)^{n+2}}\,d\theta + \left.\frac{f(\zeta(\theta))}{(\zeta(\theta) - z)^{n+1}}\right|_0^{2\pi}.$$

Note that the $\zeta'(\theta)$ that disappeared when we "antidifferentiated" $f'(\zeta(\theta))\zeta'(\theta)$ reappeared when we differentiated the denominator. The minus signs cancel. We recognize that the *integral* on the right, when multiplied by $\frac{n!}{2\pi i}$, is

$$\frac{(n+1)!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+2}}\,d\zeta \quad \text{and we sse that the term} \quad \left.\frac{f(\zeta(\theta))}{(\zeta(\theta) - z)^{n+1}}\right|_0^{2\pi} = 0, \quad \text{for} \quad \frac{f(\zeta(0))}{(\zeta(0) - z)^{n+1}} = \frac{f(\zeta(2\pi))}{(\zeta(2\pi) - z)^{n+1}}.$$

Thus (6) becomes

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+2}}\,d\zeta \quad \text{for all} \quad z \quad \text{with} \quad |z - z_o| < R, \tag{7}$$

which completes the proof of (1) and this solution.