Math 5616H, Spring 2006 **A partial solution for Special Problem 4**

Special Problem 4: Due Mar 8

How "should" we define $BV[0, +\infty)$? The problem is, what do we use as a substitute for a "partition" of $[0, +\infty)$? Is $\frac{\sin x}{x}$ in $BV[0, +\infty)$? What about $\frac{\sin^2 x}{x^2}$? Why?

- If $f \in BV[a, b]$ we have seen that $V(x) := V(f, [a, x])$ is an increasing function. Thue we "should" define

$$BV[0, +\infty) := \{f : [0, +\infty) \to \mathbb{R} : V(x) \text{ is bounded above on } [0, +\infty)\}.$$

Then, if $f \in BV[0, +\infty)$, $L := \lim_{x\to\infty} V(x)$ exists and we set $V(f, [0, +\infty)) = L$.

- Pretty much everyone found expanding partitions π_N, of $[0, \pi N]$, such that $V_{\pi_N} \to +\infty$ as $N \to \infty$. This showed $\frac{\sin x}{x} \notin BV[0, +\infty)$.

- Pretty much everyone realized that $f(x) := \frac{\sin^2 x}{x^2} \in BV[0, +\infty)$. Showing this is more difficult because we need to show that $V(x)$ is a bounded function of $x \in [0, +\infty)$.

We let $g(x) = \sin^2 x$ and $h(x) = 1/x^2$. Then $f = gh$.

On each interval $[n\pi, (n+1)\pi]$ $\sin^2 x$ increases from 0 to 1 (when $x = (n + 1/2)\pi$) and then decreases from 1 to 0 (when $x = (n+1)\pi$). Therefore $V(g, [n\pi, (n+1)\pi]) = 2$ and $g(x) \le 1$, all x.

When $n > 0$ and $\tau | [n\pi, (n+1)\pi]$

$$\text{for each } i, \quad 1 \le i \le n_\tau, \quad |f(\tau_i) - f(\tau_{i-1})| \le |g(\tau_i)||h(\tau_i) - h(\tau_{i-1})| + |g(\tau_i) - g(\tau_{i-1})||h(\tau_{i-1})|.$$

Since $g(x) \le 1$ and h is decreasing

$$|f(\tau_i) - f(\tau_{i-1})| \le |h(\tau_i) - h(\tau_{i-1})| + |g(\tau_i) - g(\tau_{i-1})|\frac{1}{(n\pi)^2}.$$

Hence $V(f, [n\pi, (n+1)\pi]) \le V(h, [n\pi, (n+1)\pi]) + V(g, [n\pi, (n+1)\pi])\frac{1}{(n\pi)^2} < \frac{3}{(n\pi)^2}$. To get the last inequality we used the overestimate $V(h, [n\pi, (n+1)\pi]) < \frac{1}{(n\pi)^2}$. Actually, $V(h, [n\pi, (n+1)\pi]) = \frac{2n+1}{n^2(n+1)^2}$ but it is pointless to use it because the other term is already pretty close to $\frac{2}{(n\pi)^2}$.

Since $\sum 1/n^2$ converges, we have shown that $f \in BV[\pi, +\infty)$. Since f decreases from 1 to 0 on $[0, \pi]$ and variations on adjacent intervals add, $f \in BV[0, +\infty)$.

Note: To show that f decreases on $[0, \pi]$ we differentiate, using the quotient rule, factor out $2\sin x \cos x$ on $(0, \pi/2)$ and use $x < \tan x$, while on $[\pi/2, \pi]$ the numerator in $f'(x)$ is negative.