Math 4606, Summer 2004: *A solution for Special Problem 4*

Special Problem 4: Due July 8

Given: there is a sequence $\{r_n\}$ such that every rational number in $(0,\,1)$ appears in $\{r_n\}$ exactly once, and every r_n is a rational number in $(0,\,1)$.

Given: For all real numbers a and b, if $a < b$ then there exists a rational number r such that $a < r < b$.

Prove that for every $\bar{x} \in (0,\,1)$, whether rational or irrational, there exists a subsequence of $\{r_n\}$ that converges to $\bar{x}$. Hint: Use the "Axiom of Continuity" Theorem.

Let $\bar{x} \in (0,\,1)$ be given. We can begin by selecting as a subsequence of $\{r_n\}$ those r_n that are in $(0,\,\bar{x})$. Why are there infinitely many? We can prove this by induction. There is a rational number in the interval $(0,\,\bar{x}/2)$ (given). For the same reason there is a rational number in *each* interval $I_n := (\frac{n-1}{n}\bar{x},\,\frac{n}{n+1}\bar{x})$. Since all these intervals are disjoint, we have found infinitely many rationals in $(0,\,\bar{x})$. These rational numbers are in increasing order, and each of them is in $\{r_n\}$ but they do not necessarily form a subsequence of $\{r_n\}$, because we have not forced their subscripts to increase strictly. Let us observe that the same argument we just did shows that each I_n contains infinitely many rationals. Let E_n be the set of all the subscripts of all the r_m's that are in I_n. In symbols,

$$E_n := \{m \in \mathbb{N} : r_m \in I_n\}. \text{ Then } E_n \text{ is infinite, hence nonempty.}$$

Here is one way you might proceed:

Let $n_1 := \min E_1$. Then $r_{n_1} \in I_1$. Next, let $n_2 := \min(E_2 \setminus [1,\,n_1])$. Since $[1,\,n_1] \cap \mathbb{N}$ is finite, $E_2 \setminus [1,\,n_1]$ is infinite, hence nonempty. Then $r_{n_2} \in I_2$ and $n_2 > n_1$. Moreover, $r_{n_2} > r_{n_1}$ because every number in I_2 is greater than every number in I_1. We continue now by induction, defining $n_{j+1} := \min(E_{j+1} \setminus [1,\,n_j])$, and we can change subscripts in the last three sentences to show that r_{n_j} is a strictly increasing subsequence of $\{r_n\}$. By construction, $\frac{j-1}{j}\bar{x} < r_{n_j} < \bar{x}$. By the Squeeze Theorem, $r_{n_j} \to \bar{x}$.

Here is another (several of you may have had this idea, but you did not get it down on paper):

Select n_1 so that $r_{n_1} \in I_1$. We can do this because we showed (in effect) that every nonempty open interval contains infinitely many rationals, so we can choose one. Since $r_{n_1} \in I_1$, $r_{n_1} < \bar{x}$ so $(r_{n_1},\,\bar{x})$ contains infinitely many rationals. At least one of these has subscript greater than n_1 because infinitely many n's cannot be less than n_1. We continue now by induction, obtaining at each step $n_{j+1} > n_j$ and $r_{n_j} \in I_j$. Then $\frac{j-1}{j}\bar{x} < r_{n_j} < \bar{x}$. By the Squeeze Theorem, $r_{n_j} \to \bar{x}$.

Extra credit will go to those who show that the sequence defined really exists, using the Recursion Theorem.

To use the Recursion Theorem in the first way to proceed, we need a set X and a function $H : X \to X$. Our set here will be the set $\mathbb{N} \times \mathbb{N}$. We define a function $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by

$$H(k,\,\ell) := (k+1,\,\min\{E_{k+1} \setminus [1,\,\ell]\}).$$

The three sentences we reused before can be changed as follows:
Since $[1,\ell] \cap \mathbb{N}$ is finite, $E_{k+1} \setminus [1,\,\ell]$ is infinite, hence nonempty. Then with $m = \min\{E_{k+1} \setminus [1,\,\ell]\}$, $r_m \in I_{k+1}$ and $m > l$. Moreover, $r_m > r_\ell$ because every number in I_{k+1} is greater than every number in I_k.

By the Recursion Theorem, there exists a unique sequence $(j,\,n_j) \in \mathbb{N} \times \mathbb{N}$ such that $(1,\,n_1) = (1,\,\min E_1)$ and, for all $n \in \mathbb{N}$, $(j+1,\,n_{j+1}) = H(j,\,n_j)$. Because the second component of $H(j,\,n_j)$ is $\min\{E_{j+1} \setminus [1,\,n_j]$ we know that $n_{j+1} > n_j$. Because $n_{j+1} \in E_{j+1}$, $r_{n_{j+1}} \in I_{j+1}$. Thus $\frac{j}{j+1}\bar{x} < r_{n_{j+1}} < \bar{x}$. By the Squeeze Theorem, $r_{n_j} \to \bar{x}$.