Special Problem 5: Due Dec 6

Find all the singularities in $\mathbb{C}$ *of* $\cot \pi z^2$ *and the residue at each singularity.*

To have a singularity we need $\sin \pi z^2 = 0$, which means that $\cos \pi z^2 \neq 0$. (Note that $\sin^2 w + \cos^2 w = 1$ for all complex w). Then $0 = 2i \sin \pi z^2 = e^{i\pi z^2} - e^{-i\pi z^2} = e^{-i\pi z^2}(e^{2\pi i z^2} - 1)$ so $\sin \pi z^2 = 0$ if and only if $e^{2\pi i z^2} = 1$ if and only if z^2 is an integer. Thus the singularities are at the square roots of the integers. This gives us singularities at $z_0 = 0$, $z_n^+ = \sqrt{n}$, $z_n^- = -\sqrt{n}$, iz_n^+ and iz_n^-, where $n \in \mathbb{Z}^+$. This gives three basic cases.
Case 0: We will use the Theorem in §66. Near z_0, let us at first write $w = \pi z^2$. Then

$$\cot \pi z^2 = \frac{\cos w}{\sin w} = \frac{1}{w} \frac{\cos w}{(\sin w / w)} = \frac{1}{w} \frac{1 - (w^2/2) + (w^4/4!) + \cdots}{1 - (w^2/3!) + (w^4/5!) + \cdots} = \frac{\varphi(z)}{z^2},$$

where $\varphi(z)$, defined by the ratio of series shown, is nonzero at zero. The ratio is analytic at zero because the denominator is not zero at $z = 0$. Moreover, the ratio is an even function of w and hence also of z, so in its series expansion about $z_o = 0 = z_0$ all the coefficients of the odd powers of z are zero. Hence $\varphi'(0) = 0$ and the residue at z_0 is 0. Note: we can also write $\varphi(z) = z^2 \cot \pi z^2 / \pi$ *near* zero but not *at* zero. However, φ has a removable singularity at 0 and we can compute $\varphi(0)$ by taking a limit, using $\lim_{z \to 0} \sin z / z = 1$. We can also compute $\varphi'(0)$ by using the difference quotient (i.e., the *definition* of derivative). The method shown is essentially "long division," and is shorter than the one just described. We have shown:

$$\operatorname*{Res}_{z=0} \left(\cot \pi z^2 \right) = 0.$$

Case Other: Now the singularity occurs when z^2 is a nonzero integer so the pole is simple, according to Theorem 2 in §69 because now $(\sin \pi z^2)' = 2\pi z \cos \pi z^2 \neq 0$. Then according to that Theorem, if ζ is a nonzero singularity, the residue is $\cos \pi z^2 / (\sin \pi z^2)' \big|_{z=\zeta} = \cos \pi \zeta^2 / (2\pi \zeta \cos \pi \zeta^2) = 1/2\pi\zeta$. We have shown: if ζ is a nonzero residue,

$$\operatorname*{Res}_{\zeta} \left(\cot \pi z^2 \right) = \frac{1}{2\pi\zeta}.$$