Math 5616H, Spring 2006 **A solution for Special Problem 5**

Special Problem 5: Due Mar 24

Suppose that $\{f_{nm}\}$ is a double sequence of complex numbers, $n \geq 1$, $m \geq 1$ and that $\{g_m\}$ is a sequence of non-negative numbers.

$$\text{In addition suppose that } \begin{cases} \text{(i)} & \sum_{m=1}^{\infty} |f_{nm}| < \infty \qquad \text{for each } n, \\ \text{(ii)} & f_m := \lim_{n\to\infty} f_{nm} \quad \text{exists for each } m, \\ \text{(iii)} & \sum_{m=1}^{\infty} g_m < \infty, \\ \text{(iv)} & |f_{nm}| \leq g_m \qquad \text{for all } n \text{ and } m. \end{cases}$$

(a) Prove that $\displaystyle\sum_{m=1}^{\infty} |f_m| < \infty$ and $\displaystyle\lim_{n\to\infty} \sum_{m=1}^{\infty} f_{nm} = \sum_{m=1}^{\infty} f_m$.

(b) Show that without (iii) and (iv), (a) need not hold, even if $f_m = 0$ for all m.

(c) Show that if $\displaystyle\sum_{m=1}^{\infty} |f_m| < \infty$ then $h(\theta) := \displaystyle\sum_{m=1}^{\infty} f_m e^{im\theta}$ is a continuous function of $\theta \in \mathbb{R}$.

Note: Please don't bother copying the problem statement in your solution!

(a): By (ii) and (iv), $|f_m| \leq g_m$. By the Comparison Test, $\displaystyle\sum_{m=1}^{\infty} |f_m|$ converges.

To prove the main part of (a) we let $\epsilon > 0$ be given. Then, using the fact that absolute convergence implies convergence, we need to show that there exists $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$,

$$\left| \sum_{m=1}^{\infty} f_{nm} - \sum_{m=1}^{\infty} f_m \right| < \epsilon.$$

Since $|f_{nm}| \leq g_m$ and $|f_m| \leq g_m$, (iii) implies that there exists M so large that for all n we have

$$\left| \sum_{m=M+1}^{\infty} f_{nm} \right| \leq \sum_{m=M+1}^{\infty} |f_{nm}| \leq \sum_{m=M+1}^{\infty} g_m < \epsilon/4 \text{ and, similarly, } \left| \sum_{m=M+1}^{\infty} f_m \right| \leq \sum_{m=M+1}^{\infty} g_m < \epsilon/4. \text{ Hence}$$

$$(1) \qquad \left| \sum_{m=1}^{\infty} f_{nm} - \sum_{m=1}^{\infty} f_m \right| \leq \left| \sum_{m=1}^{M} f_{nm} - \sum_{m=1}^{M} f_m \right| + \left| \sum_{m=M+1}^{\infty} f_{nm} - \sum_{m=M+1}^{\infty} f_m \right| \leq \left| \sum_{m=1}^{M} f_{nm} - \sum_{m=1}^{M} f_m \right| + \epsilon/2.$$

We know by (ii) that for each m there exists N_m so large that for $n \geq N_m$, $|f_{nm} - f_m| < \epsilon/2M$. Let us define $N := \max\{N_1, \ldots, N_M\}$ and assume that $n \geq N$. We can deduce from (1) and our definition of N that

$$\left| \sum_{m=1}^{\infty} f_{nm} - \sum_{m=1}^{\infty} f_m \right| \leq \sum_{m=1}^{M} |f_{nm} - f_m| + \epsilon/2 < \sum_{m=1}^{M} \frac{\epsilon}{2M} + \epsilon/2 = \epsilon, \quad \text{if } n \geq N.$$

(b): If we take $f_{nm} := \delta_{nm}$ (Kronceker delta) then (i) and (ii) hold with $\sum_{m=1}^{\infty} f_{nm} = 1$ for all n and $f_m = 0$ for all m. But then, even though $\lim_{n\to\infty} \sum_{m=1}^{\infty} f_{nm}$ and $\sum_{m=1}^{\infty} f_m$ both exist they are not equal, being 1 and 0 respectively.

(c): By Rudin's **(4.2)** (Sequences are Good Enough), we seek to show that whenever $\theta_n \to \theta$, $h(\theta_n) \to h(\theta)$. To apply (a) we define $f_{nm} := f_m e^{im\theta_n}$ and $g_m := |f_m|$. Then $f_{nm} \to f_m e^{im\theta}$, our "$f_m$" for (a), and we have $\sum_m f_{nm} = h(\theta_n)$ and $\sum_m$ "f_m" $= \sum_m f_m e^{im\theta} = h(\theta)$. We check the hypotheses for (a):

$$\sum_{m=1}^{\infty} |f_{nm}| = \sum_{m=1}^{\infty} |f_m e^{im\theta}| = \sum_{m=1}^{\infty} g_m \text{ so (i), (ii) and (iv) hold.}$$

Since $e^{im\theta_n} \to e^{im\theta}$ for each m by the continuity of the exponential function, (iii) also holds. Thus $h(\theta_n) \to h(\theta)$ and we are done.