Math 4606, Summer 2001 **Special Problem 5:** a solution

Special Problem 5: Due July 26

Let $p \in \mathbb{Z}^+$, with $p > 1$. For $A > 0$ we define $x_1 := A + 1$ and for $n \in \mathbb{Z}^+$ we define $x_{n+1} := \dfrac{1}{p}\left((p-1)x_n + \dfrac{A}{x_n^{p-1}}\right)$. Prove that $\{x_n\}$ is strictly decreasing and that $x_n^p > A$ for all $n \in \mathbb{Z}^+$. Prove that there exists $L > 0$ such that $L^p = A$. You may need to state and prove certain properties of limits.

We begin by proving that $x_n^p > A$ for all $n \in \mathbb{Z}^+$, using induction.

To prove that $x_1^p > A$ we will use Lemma 1: *For all* $p \in \mathbb{Z}^+$ *and all* $x \in \mathbb{R}$, $y \in \mathbb{R}$, *if* $x > y \geq 0$ *then* $x^p > y^p$, and prove Lemma 1 later.

We have $p > 1$ and $A + 1 > 1$, $A + 1 > A$, so $x_1^p = (A+1)^p = (A+1)^{p-1}(A+1) > 1 \cdot (A+1) > A$ (use of the Binomial Theorem was OK).

Next we show that for all $n \in \mathbb{Z}^+$, $x_n^p > A \Rightarrow x_{n+1}^p > A$. We define $t > 1$ by $x_n^p = At$. We write

$$(*) \qquad x_{n+1} = \frac{1}{p}\left((p-1)x_n + \frac{A}{x_n^{p-1}}\right) = \frac{x_n}{p}\left((p-1) + \frac{A}{x_n^p}\right) = \frac{x_n}{p}\left((p-1) + \frac{A}{At}\right) = x_n\left(\frac{p-1}{p} + \frac{1}{pt}\right)$$

Then

$$x_{n+1}^p = x_n^p\left(\frac{p-1}{p} + \frac{1}{pt}\right)^p = At\left(\frac{p-1}{p} + \frac{1}{pt}\right)^p = Ag(t),$$

where we define the function $g(t)$ below and calculate its derivative:

$$g(t) := t\left(\frac{p-1}{p} + \frac{1}{pt}\right)^p; \; g'(t) = \left(\frac{p-1}{p} + \frac{1}{pt}\right)^p + tp\left(\frac{p-1}{p} + \frac{1}{pt}\right)^{p-1}\frac{-1}{pt^2} = \left(\frac{p-1}{p} + \frac{1}{pt}\right)^{p-1}\left(\frac{p-1}{p} + \frac{1}{pt} - \frac{1}{t}\right).$$

Now $g(1) = \left(\frac{p-1}{p} + \frac{1}{p}\right)^p = 1^p = 1$, and $\left(\frac{p-1}{p} + \frac{1}{pt} - \frac{1}{t}\right) = \left(1 - \frac{1}{p} + \frac{1}{t}\left(\frac{1}{p} - 1\right)\right) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{t}\right) > 0$. The factor raised to the power $p-1$ is positive also, so $g(1) = 1$ and $g'(t) > 0$ when $t > 1$, so $x_{n+1}^p > A$.

Next we show that $\{x_n\}$ is strictly decreasing. We have, from $(*)$, and the fact that $t > 1$,

$$\frac{x_{n+1}}{x_n} = \left(\frac{p-1}{p} + \frac{1}{pt}\right) < \left(\frac{p-1}{p} + \frac{1}{p}\right) = 1.$$

Hence $\{x_n\}$ is strictly decreasing, so by the Monotone Sequence Theorem and the fact that $x_n > 0$ (by construction), there exists $L \geq 0$ such that $x_n \to L$. We need to know that $L > 0$.

To prove that $L > 0$ we will use Lemma 2: $x_n \to L$ *and* $p \in \mathbb{Z}^+ \Rightarrow x_n^p \to L^p$. We will prove Lemma 2 later.

We have $x_n^p > A$ for all $n \in \mathbb{Z}^+$, so $A \leq \lim\limits_{n\to\infty} x_n^p = L^p$. If it were true that $L = 0$, we would have $0 = 0^p \geq A > 0$, a contradiction.

Finally, we know that if a sequence converges to a non-zero limit Λ, then the sequence of reciprocals is well-defined for sufficiently large n and converges to the reciprocal of Λ. Thus the defining equations

$$x_{n+1} = \frac{1}{p}\left((p-1)x_n + \frac{A}{x_n^{p-1}}\right) \text{ converge to } L = \frac{1}{p}\left((p-1)L + \frac{A}{L^{p-1}}\right) L = \frac{L}{p}\left((p-1) + \frac{A}{L^p}\right),$$

so on multiplying thru by $\dfrac{p}{L}$ we have $p = p - 1 + \dfrac{A}{L^p}$, or $L^p = A$.

We are now able to write: $\sqrt[p]{A} := L$.

Lemma 1: *For all* $p \in \mathbb{Z}^+$ *and all* $x \in \mathbb{R}$, $y \in \mathbb{R}$, *if* $x > y \geq 0$ *then* $x^p > y^p$,

Proof: The case $p = 1$ is trivial. If $p \in \mathbb{Z}^+$ and $x > y \geq 0 \Rightarrow x^p > y^p$, then $x^{p+1} = xx^p > yx^p \geq yy^p = y^{p+1}$.

Lemma 2: $x_n \to L$ *and* $p \in \mathbb{Z}^+ \Rightarrow x_n^p \to L^p$. *Proof:* This holds by hypothesis when $p = 1$. If the Lemma holds for som $p \in \mathbb{Z}^+$, then $x_n^{p+1} - L^{p+1} = x_n^{p+1} - x_nL^p + x_nL^p - L^{p+1} = x_n(x_n^p - L^p) + (x_n - L)L^p$. We can now use the product and sum limit theorems in the right-hand-side, and this completes the proof.