Special Problem 6: Due Apr 12

This is a twenty-point Special Problem! Chapter 7, # 13.

Assume that $\{f_n\}$ *is a sequence of monotonically increasing functions on* $\mathbb{R}$ *with* $0 \le f_n(x) \le 1$ *for all* x *and* n.
(a) *Prove that there is a function* f *and a sequence* $\{n_k\}$ *such that*

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \quad \textit{for every } \; x \in \mathbb{R}.$$

(b) *If, moreover,* f *is continuous, prove that* $f_{n_k} \to f$ *uniformly on compact subsets of* $\mathbb{R}$.
(c) *(Added) Give an example to show that the convergence in* (b) *need not be uniform on the whole of* $\mathbb{R}$.

Since nearly everyone followed Rudin's listed hints this solution will follow the list too.

(i) Since the functions f_n are pointwise bounded, we can apply $(\mathbf{7.23})$ to obtain a subsequence f_{n_j} that converges pointwise on $\mathbb{Q}$ to a function $f : \mathbb{Q} \to \mathbb{R}$.

(1) **Lemma:** *If* $\{g_n\}$ *is a sequence of increasing functions on* $E \subseteq \mathbb{R}$ *that converges in* E *to a function* $g : E \to \mathbb{R}$ *then* g *is increasing on* E.

Proof: Let $x < y$, both in E. Then $g(y) - g(x) = \lim_n (g_n(y) - g_n(x)) \ge 0$.

(ii) Now we define f on $\mathbb{R}$ by $f(x) := \sup\limits_{r\le x} f(r)$, where the supremum is taken over $r \in \mathbb{Q}$. We note that this definition is *consistent* on $\mathbb{Q}$, meaning that if $s \in \mathbb{Q}$, then $\sup\limits_{r\le s} f(r) = f(s)$ (as originally defined), making $f(s)$ an upper bound for $\{f(r) : r \in \mathbb{Q}$ and $r \le s\}$ that is also in the set.

(Some people must have felt this, using something like $\tilde{f}$ instead of Rudin's f)

We now need to show that f is increasing on $\mathbb{R}$. We can't use Lemma (1) again, though. Thus suppose that $x < y$ are two real numbers. But then $f(x) \le f(y)$ because $\{f(r) : r \in \mathbb{Q}$ and $r \le x\} \subseteq \{f(r) : r \in \mathbb{Q}$ and $r \le y\}$ and the supremum of a larger set is at least as large as the supremum of a smaller one.

(iii) Let $\epsilon > 0$ be given. If f is continuous at x there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon/2$. There exist rational numbers r and s with $x - \delta < r < x < s < x + \delta$. There exists K such that $j \ge K$ implies $f(r) - \epsilon/2 < f_{n_j}(r) < f(r) + \epsilon/2$ and $f(s) - \epsilon/2 < f_{n_j}(s) < f(s) + \epsilon/2$. Let $j \ge K$. Then

$$f(x) - f_{n_j}(x) \le f(x) - f_{n_j}(r) < f(x) - (f(r) - \epsilon/2) = (f(x) - f(r)) + \epsilon/2 < \epsilon. \tag{2}$$

The first inequality holds because f_{n_j} increases: replacing x by $r < x$ means we subtract less. The next one holds because j is large enough and the last follows from our application of continuity at x. We re-do (2), using s instead of r. Now we are subtracting more (possibly) than $f_{n_j}(x)$ so we get

$$f(x) - f_{n_j}(x) \ge f(x) - f_{n_j}(s) > f(x) - (f(s) + \epsilon/2) = (f(x) - f(s)) - \epsilon/2 > -\epsilon. \tag{3}$$

Therefore $j \ge K$ implies that $|f(x) - f_{n_j}(x)| < \epsilon$.

We have proved that $f_{n_j}(x) \to f(x)$ as $j \to \infty$, if x is rational or if f is continuous at x.

(iv) We proved earlier that f increases on $\mathbb{R}$. Thus f has, at most, countably many discontinuities. We denote the set of them by D. If D is not empty, by $(\mathbf{7.23})$ there is a subsequence of f_{n_j} that converges on D to a function $g : D \to \mathbb{R}$. Of course, the subsequence converges on $\mathbb{R} \setminus D$ as well. Let us denote our subsequence, not by $\{f_{n_{j_k}}\}$ but just by $\{f_{n_k}\}$. It is *not* true, in general, that $g(x) = f(x)$ in D. Here is an example: let $f_n(x) := 1/(1+n)$ if $x < \sqrt{2}$ and let $f_n(x) := n/(1+n)$ if $x \ge \sqrt{2}$. Then $\{f_n\}$ converges pointwise to a function g that is zero if $x < \sqrt{2}$ and one if $x \ge \sqrt{2}$. Then when we define f, as before, by $f(x) := \sup\limits_{r\le x} f(r)$, $r \in \mathbb{Q}$, we have $f(\sqrt{2}) = 0 \ne 1 = \lim_n f_n(\sqrt{2})$. Therefore we must *redefine* our original f on D:

$$F(x) := \begin{cases} f(x) & \text{if } x \notin D; \\ g(x) & \text{if } x \in D. \end{cases}$$

Then, as we have shown, $f_{n_k}(x) \to F(x)$ for all $x \in \mathbb{R}$. A question remains: is F increasing. The answer is "yes," by Lemma (1), applied to $\{f_{n_k}\}$, our subsequence of a subsequence, with $E = \mathbb{R}$. This completes the proof of (a).

We next turn to proving (b). Since every compact set is contained in a compact interval it is enough to show that $f_{n_k}(x) \to F(x)$ uniformly in $[a, b]$ where a and b are real numbers and $a < b$. We will also change the name $F(x)$ to $f(x)$. Indeed, in this case we did not replace $\{f_{n_j}\}$ by a subsequence. We will still call it $\{f_{n_k}\}$.

Thus, given $\epsilon > 0$ we need to prove that there exists K such that $k \ge K$ implies that $|f_{n_k}(x) - f(x)| < \epsilon$ for all $x \in [a, b]$. As f is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that for all x, y in $[a, b]$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon/4$. We next select a positive integer M such that $(b - a)/M < \delta$ and construct the partition $\pi\big|[a, b]$ given by $x_i := a + i\frac{b-a}{M}$, $0 \le i \le M$.

For each of our x_i there is a K_i such that $k \ge K_i \Rightarrow |f_{n_k}(x_i) - f(x_i)| < \epsilon/2$. We thus set $K := \max\{K_1, \ldots, K_M\}$, so that for each i, $0 \le i \le M$, $k \ge K \Rightarrow |f_{n_k}(x_i) - f(x_i)| < \epsilon/2$.

We can now show that for all $x \in [a, b]$, $k \ge K \Rightarrow |f_{n_k}(x) - f(x)| < \epsilon$. For definiteness, given $x \in [a, b]$ we select the *first* i such that $x \in I_i$. Then since $f_{n_k}(x) - f_{n_k}(x_i) \le f_{n_k}(x_i) - f_{n_k}(x_i) = 0$ and $|x - x_i| < \delta$,

$$f_{n_k}(x) - f(x) = f_{n_k}(x) - f_{n_k}(x_i) + f_{n_k}(x_i) - f(x_i) + f(x_i) - f(x) \le f_{n_k}(x_i) - f(x_i) + f(x_i) - f(x) < \epsilon.$$

Similarly,

$$f_{n_k}(x) - f(x) \ge f_{n_k}(x_{i-1}) - f(x_{i-1}) + f(x_{i-1}) - f(x) > -\epsilon.$$

It follows (**Definition 7.7**) that $f_{n_k}(x) \to f(x)$ uniformly in $[a, b]$.

Here are some details for “similarly:” since $f_{n_k}(x) - f_{n_k}(x_{i-1}) \ge f_{n_k}(x_{i-1}) - f_{n_k}(x_{i-1}) = 0$ and $|x - x_{i-1}| < \delta$,

$$f_{n_k}(x) - f(x) = f_{n_k}(x) - f_{n_k}(x_{i-1}) + f_{n_k}(x_{i-1}) - f(x_{i-1}) + f(x_{i-1}) - f(x) \ge f_{n_k}(x_{i-1}) - f(x_{i-1}) + f(x_{i-1}) - f(x) > -\epsilon.$$

(c) Here is an example showing that convergence to a continuous f need not be uniform on $\mathbb{R}$. We let

$$f_n(x) := \begin{cases} 0 & \text{if } x \le n; \\ 1 & \text{if } x > n. \end{cases}$$

Then $f_n(x) \to 0$ for all x, but (see $(\mathbf{7.9})$) $M_n := \sup_x |f_n(x) - f(x)| = 1 \not\to 0$ as $n \to \infty$. Thus the convergence is not uniform on $\mathbb{R}$.