Special Problem 6: Due July 19

This one is worth 15 points! Prove the *Sequences are good enough for continuity* Theorem: *Given that* $f : D \to \mathbb{R}$ *and* $x_o \in D$, f *is continuous at* x_o *if and only if for every sequence* $\{x_n\}$ *such that* $x_n \in D$ *for all* n *and such that* $x_n \to x_o$, $f(x_n) \to f(x_o)$ [9]. *State a corresponding result – Sequences are good enough for limits – for "f has a limit as* $x \in D \to x_o$" [6]. You should cite a Special Problem to prove half of the first result! Caution: the "corresponding result" is not a trivial modification of the first one! No proof is required, but you should try to prove your statement "in your head" to be sure it's right. You might want to use $f(x) := (x^3 - 1)/(x^2 - 1)$, with $x_o = 1$, as an example.

If f is continuous at x_o, the given hypotheses satisfy the conditions in Special problem 3, so " $\Rightarrow$ " is true.

If the sequential condition holds, suppose f is NOT continuous at x_o. This means that there exists $\epsilon > 0$ such that for all $\delta > 0$ there exist $x \in D$ such that $|x - x_o| < \delta$ but $|f(x) - f(x_o)| \geq \epsilon$.

Take $\delta_n = 1/n$, and choose $x_n \in D$ such that $|x_n - x_o| < 1/n$ but $|f(x_n) - f(x_o)| \geq \epsilon$. Then $x_n \to x_o$, but $f(x_n)$ does not converge to $f(x_o)$, which contradicts our hypothesis. Thus f is continuous at x_o.

The corresponding result: *If* $f : D \to \mathbb{R}$ *and there exists* $L \in \mathbb{R}$ *such that for every sequence* $\{x_n\}$ *with every* $x_n \in D$ *and with* $x_n \neq x_o$ *for every* n, *such that* $\{x_n\}$ *converges to* x_o, *it is true that* $f(x_n) \to L$. *Then* $f(x) \to L$ *as* $x \to x_o$ *with* $x \in D$. *Conversely, if* $f(x) \to L$ *as* $x \to x_o$ *with* $x \in D$ *then* $f(x_n) \to L$ *whenever* $x_n \in D$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x_o$ *and* $x_n \neq x_o$ *for all* $n \in \mathbb{N}$.