Special Problem 7: Due Apr 26

Find a formula (in terms of its entries) for $\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|$, where the entries are *real* numbers. Use no eigenvalues!

Our *definition* of $\|A\|$, when A is an $m \times n$ matrix, is

$$\|A\| := \sup_{|x| \le 1} |Ax|.$$

It will be more convenient to work with $\sup_{|x|=1} |Ax| \le \|A\|$, so we show now that $\|A\| = \sup_{|x|=1} |Ax|$. Since $\|A\| = 0$ if and only if $A = 0$, the desired version is true in this case. If $A \ne 0$ then $|Ax| > 0$ for some x, this x is necessarily non-zero and $\|A\| > 0$. Now there exists $\{x_k\}$, with each $|x_k| \le 1$, such that $|Ax_k| \to \|A\|$. We may assume that each $x_k \ne 0$ (work with k so large that $|Ax_k| > \|A\|/2$). But then

$$|Ax_k| = |x_k| \left| A\left(\frac{x_k}{|x_k|}\right) \right| \le \left| A\left(\frac{x_k}{|x_k|}\right) \right| \le \sup_{|x|=1} |Ax|, \quad \text{so} \quad \|A\| = \lim_{k\to\infty} |Ax_k| \le \sup_{|x|=1} |Ax|.$$

Thus we have $\|A\| = \sup_{|x|=1} |Ax|$.

In $\mathbb{R}^2$ each x with $|x| = 1$ can be expressed as $x = (\cos\theta, \sin\theta)$ with $\theta \in [0, 2\pi]$. Thus

$$\begin{aligned} \|A\|^2 &= \sup_{|x|=1} |Ax|^2 = \sup_{\theta \in [0,\, 2\pi]} \{(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2\} \\ &= \max_{\theta \in [0,\, 2\pi]} \{(a^2 + c^2)\cos^2\theta + 2(ab + cd)\cos\theta\sin\theta + (b^2 + d^2)\sin^2\theta\}. \end{aligned} \tag{1}$$

For brevity let us write $E := a^2 + c^2$, $F := ab + cd$ and $G := b^2 + d^2$. We will rewrite the expression being maximized in the last line of (1) in terms of E, F and G, use double-angle formulas to replace squares and products of cosine and sine (of the same angle) by cosine or sine of the doubled angle, then recognize the opportunity to use polar coordinates to express the coefficients of those double angles. Another application of the formula for the cosine of a sum will then lead to an expression very easy to maximize:

$$\begin{aligned} E\cos^2\theta + 2F\cos\theta\sin\theta + G\sin^2\theta &= E\frac{1+\cos 2\theta}{2} + F\sin 2\theta + G\frac{1-\cos 2\theta}{2} \\ &= \frac{E+G}{2} + \frac{E-G}{2}\cos 2\theta + F\sin 2\theta \\ &=: K + L\cos 2\theta + M\sin 2\theta \\ &= K + \sqrt{L^2+M^2}\left(\frac{L}{\sqrt{L^2+M^2}}\cos 2\theta + \frac{M}{\sqrt{L^2+M^2}}\sin 2\theta\right) \\ &= K + \sqrt{L^2+M^2}\,(\cos\delta\cos 2\theta + \sin\delta\sin 2\theta) \\ &= K + \sqrt{L^2+M^2}\,\cos(\delta - 2\theta). \end{aligned}$$

Here, $K := (E+G)/2$, $L := (E-G)/2$ and $M := F$. Since K and $\sqrt{L^2+M^2}$ are non-negative we see that the maximum of $K + \sqrt{L^2+M^2}\cos(\delta - 2\theta)$ is

$$K + \sqrt{L^2+M^2} = \frac{a^2+b^2+c^2+d^2}{2} + \sqrt{\left(\frac{a^2-b^2+c^2-d^2}{2}\right)^2 + (ab+cd)^2}\,.$$

Thus

$$\|A\| = \sqrt{\frac{a^2+b^2+c^2+d^2}{2} + \sqrt{\left(\frac{a^2-b^2+c^2-d^2}{2}\right)^2 + (ab+cd)^2}}\,.$$