Special Problem 7 Let us define a space into which we can embed all the $\mathbb{R}^k$ and $\mathbb{C}^k$ in a natural way: we set ℓ^2 equal to the collection of all complex sequences $x = \{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. This is usually spoken of as “little ell 2.” When we want to use this to embed spaces $\mathbb{R}^k$ we can just use real-valued sequences, and we might say “*real* little ell 2.” The embeddings are these: $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{R}^k$ is transformed into the sequence whose first k terms are the v_j, $j = 1, \ldots, k$, and whose terms v_j with $j > k$ are all zero. In ℓ^2 we define the “standard orthonormal basis” to be the set of sequences e_n, $n = 1, 2, \ldots$ where e_n has all terms zero except the n-th term, which is 1. We can then express $x \in \ell^2$ as the (formal) sum $x = \sum_{n=1}^{\infty} x_n e_n$. The *norm* of $x \in \ell^2$ is defined to be $||x|| := \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$, and the *inner product* of $x \in \ell^2$ and $y \in \ell^2$ is defined by $\langle x, y\rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$.
(a) State and prove the Schwarz inequality for ℓ^2;
(b) Show that $d(x, y) := ||x - y||$ is a metric on ℓ^2;
(c) Starting with $\mathbf{v}_1 := e_1$, construct *the* sequence of vectors $\{\mathbf{v}_k\}_{k=1}^{\infty}$ in ℓ^2 with these properties:

$$\begin{aligned} &\text{(i)} \quad ||\mathbf{v}_k|| = 1 \quad \text{for all} \quad k; \\ &\text{(ii)} \quad \langle \mathbf{v}_k, e_k\rangle > 0 \quad \text{for all} \quad k; \\ &\text{(iii)} \quad \langle \mathbf{v}_k, e_j\rangle = 0 \quad \text{for all} \quad j > k; \\ &\text{(iv)} \quad \langle \mathbf{v}_j, \mathbf{v}_k\rangle = \frac{1}{2} \quad \text{for all} \quad j \neq k. \end{aligned}$$

Show that this sequence of vectors satisfies $d(\mathbf{v}_j, \mathbf{v}_k) = 1$ for all $j \neq k$.
You might assign youself these problems: Show that ℓ^2 is a complete metric space; show that the property that every pair of unequal vectors are distance 1 apart can only hold in $\mathbb{R}^k$ for sets up to a certain size that depends on k, and find that size. Hint: assume that one of the vectors in the set is the vector $\mathbf{0}$.

A Solution

(a) **The Schwarz Inequality** *For all* x *and* y *in* ℓ^2,

$$(\mathbf{S}) \qquad |\langle x, y\rangle| \le \|x\|\|y\|,$$

and equality holds if and only if one of x *and* y *is a scalar multiple of the other.*

Proof: To even begin, we first must verify that $\langle x, y\rangle$ makes sense for all x and y in ℓ^2. It will be enough to show that the series $\sum_{n=1}^{\infty} x_n\overline{y_n}$ converges absolutely. Since $|x_n\overline{y_n}| = |x_n||\overline{y_n}| = |x_n||y_n|$ and since $|x_n|^2 - 2|x_n||y_n| + |y_n|^2 = (|x_n| - |y_n|)^2 \ge 0$,

$$(1) \qquad \left|\sum_{n=1}^{\infty} x_n\overline{y_n}\right| \le \sum_{n=1}^{\infty} |x_n\overline{y_n}| \le \sum_{n=1}^{\infty} \frac{|x_n|^2 + |y_n|^2}{2} < \infty.$$

If at least one of x and y is $\mathbf{0}$, both sides of the inequality $(\mathbf{S})$ are zero, so the inequality holds, and we have equality. The first one of x and y that is $\mathbf{0}$ is then equal to the scalar zero times the other one. Thus the complete result holds in this case.

If we are not in the case just discussed, neither of $\|x\|$ and $\|y\|$ can be zero.

Let us consider the very special case $\|x\| = 1 = \|y\|$. The right-hand side of inequality (1) is then $1 = \|x\|\|y\|$, which shows that $(\mathbf{S})$ holds in this very special case.

In the case when neither of x and y is $\mathbf{0}$, we can define $x' := x/\|x\|$ and define $y' := y/\|y\|$, and then since $\|x'\| = 1 = \|y'\|$,

$$|\langle x', y'\rangle| = \left|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right| \le 1.$$

When we now multiply both sides of the second inequality by $\|x\|\|y\|$, we have $(\mathbf{S})$ in all cases.

It remains to prove the statement about equality in the case when neither x nor y is $\mathbf{0}$. Thus we suppose $|\langle x, y\rangle| = \|x\|\|y\| \neq 0$. Let us define the “sign” of a complex number z to be $\operatorname{sgn}(z) := z/|z|$ if $z \neq 0$, and define

$\operatorname{sgn}(0) := 0$. Except when $z = 0$, $\operatorname{sgn}(z)$ is a complex number whose absolute value is 1. We will exploit this, and also the fact that for all complex z, $|z| = z\,\overline{\operatorname{sgn}(z)}$. Let us abbreviate by setting $\omega := \operatorname{sgn}(\langle x,\, y\rangle)$, and observe that $\overline{\omega}\langle x,\, y\rangle = |\langle x,\, y\rangle| = \|x\|\|y\|$ in our case under study. Next we define a function $P(t)$ of the real variable t by:

$$P(t) := \|x - t\omega y\|^2 = \langle x - t\omega y,\, x - t\omega y\rangle = \sum_{n=1}^{\infty}(x_n - t\omega y_n)\overline{(x_n - t\omega y_n)}.$$

We wish to show that $P(t_o) = 0$ for some t_o, which will show that every term in the sequence $x - t_o\omega y$ is zero, or that $x = (t_o\omega)y$, as desired. All we need to do is expand, noting that $z + \bar{z} = 2\mathrm{Re}(z)$ and that $|\omega|^2 = 1$:

$$\begin{aligned}
P(t) &= \sum_{n=1}^{\infty}\left(|x_n|^2 - x_n\overline{t\omega y_n} - t\omega y_n\overline{x_n} + t^2|y_n|^2\right)\\
&= \sum_{n=1}^{\infty}\left(|x_n|^2 - 2\mathrm{Re}(x_n\overline{t\omega y_n}) + t^2|y_n|^2\right)\\
&= \sum_{n=1}^{\infty}|x_n|^2 - 2t\sum_{n=1}^{\infty}\mathrm{Re}(x_n\overline{\omega y_n}) + t^2\sum_{n=1}^{\infty}|y_n|^2\\
&= \|x\|^2 - 2t\mathrm{Re}\left(\overline{\omega}\sum_{n=1}^{\infty}x_n\overline{y_n}\right) + t^2\|y\|^2\\
&= \|x\|^2 - 2t\mathrm{Re}\left(\overline{\omega}\langle x,\, y\rangle\right) + t^2\|y\|^2\\
&= \|x\|^2 - 2t\|x\|\|y\| + t^2\|y\|^2 = (\|x\| - t\|y\|)^2 = 0
\end{aligned}$$

when $t_o = t = \|x\|/\|y\|$. This completes this solution of (a), which has included a lot of review! 4U: line 3 to line 4?

(b) The proof of (b) is the same as it was in the case of $\mathbb{C}^k$, because we now have the Schwarz inequality available, and we know that the inner product makes sense (both shown in (a) using properties of absolutely convergent series).

(c) This has complicated logic! We will begin by *assuming* that a sequence satisfying (i)–(iv) exists, which will give *necessary* conditions. Once those conditions are in hand, we will be able to show that they suffice as well, and that the coefficients that are necessary actually exist (which will happen in the course of the investigation). We are given that $v_1 = e_1$. Thus, if we write

$$v_n = \sum_{k=1}^{\infty}v_{nk}e_k, \quad \left(= \sum_{k=1}^{n}v_{nk}e_k \text{ by (iii)} \right) \tag{2}$$

condition (iv) gives $\frac{1}{2} = \langle v_n,\, v_1\rangle = v_{n1}$ for all $n > 1$. This tells us that $v_2 = \frac{1}{2}e_1 + v_{22}e_2$, whereupon (i) tells us that $\frac{1}{4} + v_{22}^2 = 1$, or that, because of (ii), $v_{22} = \frac{\sqrt{3}}{2}$. That is, $v_2 = \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$. You have, no doubt, carried these steps out up to $n = 4$ or 5, and you have seen that the following statement can probably be proved by induction:

If there exists a sequence $\{v_n\}$ in ℓ^2 that satisfies conditions (i)–(iv), then there exist sequences $\{\beta_k\}$ and $\{\alpha_n\}$ such that for all integers $n > 1$,

$$\begin{aligned}
v_n &= \alpha_n\, e_n + \sum_{k=1}^{n-1}\beta_k\, e_k;\\
\alpha_n^2 &> 1/2;\\
\beta_k &> 0.
\end{aligned} \tag{\textbf{T}}$$

We can set up a statement that depends on n and prove that by induction. We will use the parenthesized formula in (2) to represent v_n, and define our statement $P(n)$ as follows:

$P(n) :=$ "there exist $\beta_k > 0$, $1 \le k < n$, and $\alpha_k > 1/\sqrt{2}$, $1 \le k \le n$, such that $v_{kk} = \alpha_k$, and for all $m \in \mathbb{N}$,

and for all k, if $1 \le k < m$ and $k < n$, then $v_{mk} = \beta_k$."

We may need to take a little care to make sure that the sequences $\{\beta_k\}$ and $\{\alpha_k\}$ don't depend on n. We have shown that $P(1)$ and $P(2)$ are true, with $\alpha_1 = 1$, $\alpha_2 = \sqrt{3}/2$ and $\beta_1 = 1/2$. Suppose that $P(n)$ is true for some $n > 1$. Then, for every $m > n$, $v_{mk} = \beta_k$ if $k < n$, and $v_n = \sum_{k=1}^{n-1} \beta_k e_k + \alpha_n e_n$. Thus

$$\frac{1}{2} = \langle v_m, v_n \rangle = \sum_{k=1}^{n-1} \beta_k^2 + v_{mn}\alpha_n = 1 - \alpha_n^2 + v_{mn}\alpha_n,$$

so $v_{mn} = (\alpha_n^2 - \frac{1}{2})\alpha_n^{-1}$ is independent of m for all $m > n$, so we can define $\beta_n := (\alpha_n^2 - \frac{1}{2})\alpha_n^{-1} > 0$. Then we set α_{n+1} equal to $v_{n+1,n+1}$, getting

$$1 = \|v_{n+1}\|^2 = \sum_{k=1}^{n} \beta_k^2 + \alpha_{n+1}^2, \quad \text{so} \quad \alpha_{n+1}^2 = 1 - (1 - \alpha_n^2) - \beta_n^2 = \alpha_n^2 - (\alpha_n^2 - \frac{1}{2})^2 \alpha_n^{-2}.$$

Hence $\{\alpha_n^2\}$ satisfies a recursive equation, and β_n can be defined in terms of α_n. The recursive equation can be simplified quite a bit:

$$\alpha_{n+1}^2 = \alpha_n^2 - (\alpha_n^2 - \frac{1}{2})^2\alpha_n^{-2} = \big((\alpha_n^2)^2 - (\alpha_n^2 - \frac{1}{2})^2\big)\alpha_n^{-2} = \big(\frac{1}{2}(2\alpha_n^2 - \frac{1}{2})^2\big)\alpha_n^{-2} = \big(\frac{1}{2}(2\alpha_n^2 - \frac{1}{2})^2\big)\alpha_n^{-2}, \quad \text{so}$$

$$\alpha_{n+1}^2 = \big(\frac{1}{2}(2\alpha_n^2 - \frac{1}{2})^2\big)\alpha_n^{-2} = \frac{4\alpha_n^2 - 1}{4\alpha_n^2} = 1 - \frac{1}{4\alpha_n^2}.$$

Since $\alpha_n^2 > 1/2$, we can write $\alpha_n^2 = p + 1/2$, where $p > 0$, and thus

$$\alpha_{n+1}^2 = 1 - \frac{1}{4\alpha_n^2} = 1 - \frac{1}{4(p + 1/2)} = 1 - \frac{1}{2 + 4p} > \frac{1}{2}.$$

What we have shown is that there exist $\beta_k > 0$, $1 \le k < n+1$, and $\alpha_k > 1/\sqrt{2}$, $1 \le k \le n+1$, such that $v_{kk} = \alpha_k$, and for all $m \in \mathbb{N}$,

$$\text{for all} \quad k, \quad \text{if} \ 1 \le k < m \ \text{ and } \ k < n+1, \ \text{ then } \ v_{mk} = \beta_k.$$

That is, we have shown that $P(n) \Rightarrow P(n+1)$. We have therefore shown that if the sequence $\{v_n\}$ satisfies (i)–(iv), then the terms of each v_n can be expressed in terms of certain positive numbers β_k and α_n. We have implicitly proved that the β_k and α_n exist. Let us now (as review) show this independently of the existence of the sequence $\{v_n\}$. To emphasize this, we'll work with $A_n = \alpha_n^2$.

Given that $A_1 = 1$, and that we seek to define subsequent A_n by

$$A_{n+1} = 1 - \frac{1}{4A_n},$$

we can succeed by showing that $A_n > 1/2$ for each n, which shows that the sequence is well-defined. But if $A_n > 1/2$, then $A_n = p + 1/2$, and the argument we used before shows that $A_{n+1} > 1/2$. We can show more:

$$A_n - A_{n+1} = A_n - 1 + \frac{1}{4A_n} = \frac{4A_n^2 - 4A_n + 1}{4A_n} = \frac{(2A_n - 1)^2}{4A_n} > 0. \tag{3}$$

Thus $\{A_n\}$ is strictly decreasing, bounded below by $1/2$, hence converges. Then the left side of (3) converges to 0, and thus $A_n \to 1/2$.

We can now define $\alpha_n := \sqrt{A_n} > 1/\sqrt{2}$, and $\beta_n := (\alpha_n^2 - \frac{1}{2})\alpha_n^{-1} > 0$. With $\mathbf{v}_n = e_1$, and, for $n > 1$,

$$\mathbf{v}_n := \sum_{k=1}^{n-1} \beta_k e_k + \alpha_n e_n,$$

the sequence will satisfy (as we have, in effect, shown) (i)–(iv). You should verify this to complete the proof of (c). It is only a matter of checking (i)–(iv) for our new sequence.

Remark It has been noticed by one of you that the A_n are rational numbers with interesting properties. Perhaps you will be able to find a formula for the A_n, but this was not necessary for solving the problem!

(d) If $j \ne k$ then $\|\mathbf{v}_j - \mathbf{v}_k\|^2 = 2 - 2\langle \mathbf{v}_j, \mathbf{v}_k \rangle = 1$, as desired.

Show that ℓ^2 is a complete metric space

Suppose that $\{\mathbf{v}_n\}$ is a Cauchy sequence in ℓ^2. Then for each n, and by the Schwarz inequality,

$$|\langle \mathbf{v}_j, e_n\rangle - \langle \mathbf{v}_k, e_n\rangle| = |\langle \mathbf{v}_j - \mathbf{v}_k, e_n\rangle| \le \|\mathbf{v}_j - \mathbf{v}_k\| \to 0$$

as j and k both tend to infinity, so that $\{v_{jn}\} := \{\langle \mathbf{v}_j, e_n\rangle\}$ is a Cauchy sequence of complex numbers, hence converges to a limit x_n. This is so for each n. This gives us a sequence $\{x_n\}$ that is a natural candidate for the limit: $\mathbf{x} := (x_1, x_2, \ldots, x_n, \ldots)$.

First we show that $\mathbf{x} \in \ell^2$.
Since $\{\mathbf{v}_n\}$ is a Cauchy sequence, $\{||\mathbf{v}_k||\}$ is bounded (why? this is review). This means that there exists $M < \infty$ such that $\sum_{j=1}^{\infty} |v_{kj}|^2 \le M$, independent of k. Thus for each fixed N, $\sum_{j=1}^{N} |v_{kj}|^2 \le M$. We now let $k \to \infty$, and can conclude that $M \ge \sum_{j=1}^{N} |v_{kj}|^2 \to \sum_{j=1}^{N} |x_j|^2$. But since M is independent of N, $\sum_{j=1}^{\infty} |x_j|^2$ converges (why? this is review). Thus $\mathbf{x} \in \ell^2$.

Next we show that $\|\mathbf{v}_j - \mathbf{x}\| \to 0$.
Given $\epsilon > 0$, there exists $L \in \mathbb{Z}$ such that if j and k are both larger than L, $\|\mathbf{v}_j - \mathbf{v}_k\| < \epsilon$. Then, for any $N > 1$, $\sum_{n=1}^{N} |v_{jn} - v_{kn}|^2 < \epsilon^2$ as well. Now in this sum we let $k \to \infty$. This gives $\sum_{n=1}^{N} |v_{jn} - x_n|^2 \le \epsilon^2$, independent of N. Thus $\|\mathbf{v}_j - \mathbf{x}\| \le \epsilon$. This completes the proof of completeness.

Show that the property that every pair of unequal vectors are distance 1 apart can only hold in $\mathbb{R}^k$ for sets up to a certain size that depends on k, and find that size. Hint: assume that one of the vectors in the set is the vector $\mathbf{0}$.

Let $T \subseteq \mathbb{R}^k$ be a set that has the property ED if $x \in T$ and $y \in T$ and $x \ne y$ implies $\|x - y\| = 1$. We may assume that $0 \in T$, by subtracting one of the vectors in the set from each element in the set. Such sets exist: the set consisting of 0 and the first k elements of the sequence $\{v_n\}$ from Special problem 7(c) has $k + 1$ elements and has property ED.

Claim *No set in $\mathbb{R}^k$ that has property ED can have more than $k + 1$ elements.* Since the solution I have in mind involves the construction of an orthonormal matrix that maps T *into* the set consisting of 0 and the first k elements of the sequence $\{v_n\}$ from Special problem 7(c), I will not ask anything about this question on the Final, and will not give the construction here.

Remark on the sequence $\{A_n\}$. One of you solved the recursion

$$A_{n+1} = 1 - \frac{1}{4A_n}, \quad \text{with} \quad A_1 = 1!$$

The formula is $A_n = \dfrac{n+1}{2n}$.

$\alpha_n = \sqrt{A_n} = \sqrt{\dfrac{n+1}{2n}}$.

$\beta_n = (\alpha_n^2 - \frac{1}{2})\alpha_n^{-1} = \dfrac{1}{2n\alpha_n} = \sqrt{\dfrac{1}{2n(n+1)}}$.